Lecture: Financial Modelling

– Preparation knowledge

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Outline

- The time value of money is the greater benefit of receiving money now rather than later. It is the so-called time preference.
- The principle of the time value of money explains why interest is paid or earned: **Interest**, whether it is on a bank deposit (or debt), compensates the depositor (or lender) for the time value of money.
- It also underlies investment: investors are willing to forgo spending their money now, if they expect a favorable return on their investment in the future.

- *Simple interest* is calculated only on the *principal* amount.

- Suppose FV is the *final value*, B is the initial balance, i is the interest rate, n is the number of time periods, then, we have

$$FV = B(1 + i \times n).$$

- *Compound interest* includes interest earned on the interest which was previously accumulated.
- Suppose *FV* is the *final value*, *B* is the initial balance, *i* is the interest rate, *n* is the number of time periods, then,we have

$$FV = B(1+i)\times(1+i)\times....(1+i) = B(1+i)^n.$$

Suppose an investor deposits 10,000 yuan in the bank account for 5 years, the annal interest rate is 2.5%.

- Simple interest:

$$FV = 10,000 \times (1 + 2.5\% \times 5) = 11,250.$$

- Compound interest:

$$FV = 10,000 \times (1 + 2.5\%)^5 = 11,314.$$

- As n, the number of compounding periods per year, increases to infinity, we have the limiting case known as *continuous compounding*, i.e.,

$$\lim_{n\to\infty} B\left(1+\frac{r}{n}\right)^{nt} = Be^{rt}.$$

- Final value at time T is

$$FV_T = PV_t \times e^{r(T-t)}, \qquad T > t.$$

- Present value at time t is

$$PV_t = FV_T \times e^{-r(T-t)}, \qquad T > t.$$

- **Volatility** (often denoted by symbol σ) is the degree of variation of a asset price (or return) series over time.
- Volatility is an important factor in options trading:
  - Black-Scholes (BS) option pricing formula.
- Volatility has many other financial applications:
  - Calculating value at risk (VaR) in risk management;
  - Asset allocation under the mean-variance framework.

Since asset price (return) is a stochastic process, the actual volatility is always an unknown (unobservable). We only can get its estimations from historical observations based on e.g. some financial time-series models:

- *Moving average* (MA) model;
- *Generalized autoregressive conditional heteroskedasticity* (GARCH) model (Engle, 1982; Bollerslev, 1986).

We may get its estimations from e.g. moving average models based on M observations:

- Equally-weighting scheme:

$$\sigma_t{}^2 = \frac{1}{M-1} \times \sum_{i=1}^{M} \left(r_{t-i} - \frac{\sum_{j=1}^{M} r_{t-j}}{M} \right)^2 ;$$

- General weighting scheme:

$$\sigma_t{}^2 = \frac{1}{M-1} \times \sum_{i=1}^{M} \omega_{t-i} \left(r_{t-i} - \frac{\sum_{j=1}^{M} r_{t-j}}{M} \right)^2 .$$

- If an *autoregressive moving average* (ARMA) model is assumed for the error variance, then, the model is a *generalized autoregressive conditional heteroskedasticity* (GARCH) model.

- The *GARCH* (p, q) *model*, following the notation of original paper (Bollerslev, 1986), is given by

$$
\begin{aligned}
y_t &= x_t' b + \varepsilon_t, \qquad \varepsilon_t \mid \psi_{t-1} \sim \mathrm{N}\left(0,\ \sigma_t^2\right); \\
\sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2,
\end{aligned}
$$

where p is the order of the GARCH terms σ^2, q is the order of the ARCH terms ε^2.

1. Stochastic process
2. Markov process
3. Martingale process
4. Brownian motion
5. Itô process and Itô's lemma

Random variable:

- A random variable $X : \Omega \to E$ is a measurable function from a set of all possible outcomes Ω to a measurable space E.
- Usually X is real-valued, i.e., $E \subseteq \mathbb{R}$.

Stochastic process:

- For a given probability space and a measurable space, $(\Omega, \mathcal{F}, \mathcal{P})$, a stochastic process X is a collection of real-valued random variables, indexed by some set $\mathbb{T}$, which can be written as
$$\{X(\omega, t) : t \in \mathbb{T}\},$$
or,
$$\{X(t) : t \in \mathbb{T}\}.$$
- Stochastic process $\{X(t)\}_{t\in\mathbb{T}}$ is a binary function defined on $\Omega \times \mathbb{T}$.

- A **Markov process** is a stochastic process that has *Markov property*, which means the next value of the Markov process depends on the current (most recent) value, but it is conditionally independent of all previous values of the stochastic process:

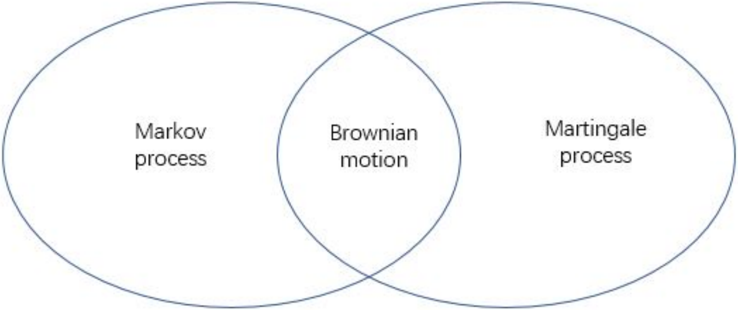
$$\begin{aligned} & \Pr\left\{X(t_n) \leq x_n \mid X(t_1) = x_1, X(t_2) = x_2, \cdots, X(t_{n-1}) = x_{n-1}\right\} \\ = \quad & \Pr\left\{X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1}\right\}. \end{aligned}$$

- A **martingale process** is a stochastic process with the property that, the expectation of the next value of a martingale is equal to the current value given all the previous values of the process, i.e.,

$$\mathbb{E}\left[X(t) \mid X(s)\right] = X(s), \qquad s < t.$$

- Martingales mathematically formalize the idea of a fair game, and they were originally developed to show that it is not possible to win a fair game on average.

Markov process v.s. martingale process



- *Brownian motion* (BM) is named after Robert Brown, a British botanist, who observed the random movement of pollen grains in water.

- Brownian motion was analysed mathematically by the American mathematician Norbert Wiener, so is also called a *Wiener process*.

- As early as 1900, Louis Bachelier, in his thesis *Theory of Speculation*, proposed Brownian motion as a model for the fluctuations of stock prices.

The definition of a standard **Brownian motion** $W(t)$ or W_t:

1. The path is continuous.
2. The path increments follow a normal distribution, i.e., for any time $t \geq 0$ and time period $\Delta t \geq 0$,

   $$\Delta W(t) \ := \ W(t+\Delta t) - W(t) \ = \ \varepsilon_i \sqrt{\Delta t}, \qquad \varepsilon_i \sim \mathrm{N}(0,1),$$

   with zero initial value $W(0) = 0$, which implies that

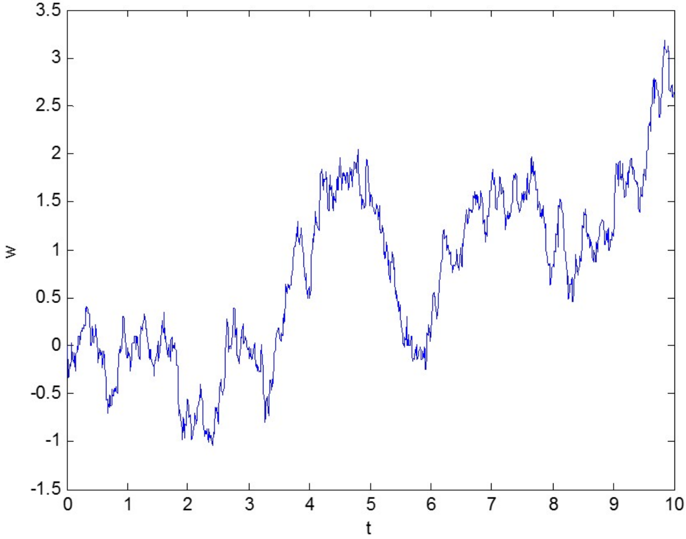
   $$\Delta W(t) \ \sim \ \mathrm{N}(0, \Delta t),$$

   or,

   $$W(T) - W(t) \ \sim \ N\left(0, T-t\right), \qquad T \geq t \geq 0.$$

3. The path increments are independent.

These are necessary and sufficient conditions for a process to be identified as a standard Brownian motion.


3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
w
0
1
2
3
4
5
6
7
8
9
10
t

- $W(t)$ is a Markov process, since for any $s, t \geq 0$,

$$\begin{aligned} & \Pr\{W(s+t) \leq a \mid W(s) = x, W(u), 0 \leq u \leq s\} \\ = \; & \Pr\{W(s+t) - W(s) \leq a - x \mid W(s) = x, W(u), 0 \leq u \leq s\} \\ = \; & \Pr\{W(s+t) - W(s) \leq a - x \mid W(s) = x\} \\ = \; & \Pr\{W(s+t) \leq a \mid W(s) = x\}. \end{aligned}$$

- $W(t)$ is a martingale process, since for any $t \geq s \geq 0$,

$$\begin{aligned} & \mathbb{E}[W(t) \mid W(s)] \\ = \; & \mathbb{E}\left[W(s) + \left(W(t) - W(s)\right) \mid W(s)\right] \\ = \; & \mathbb{E}[W(s)] + \mathbb{E}[W(t) - W(s) \mid W(s)] \\ = \; & W(s). \end{aligned}$$

A *normal Brownian motion* $X(t)$ is defined by the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = a\mathrm{d}t + b\mathrm{d}W(t),$$

or,

$$X(t) = X(0) + at + bW(t),$$

where

- $a\mathrm{d}t$ is a deterministic term, a is *drift rate*;
- $b\mathrm{d}W(t)$ is a random term, b is *volatility*;
- $X(0)$ is the *initial value* of $X(t)$ at time 0.

More generally, $X(t)$ is a *Itô process* if it satisfies the SDE

$$\mathrm{d}X(t) = a\,(X, t)\,\mathrm{d}t + b\,(X, t)\,\mathrm{d}W(t),$$

or,

$$X\,(t) = X\,(0) + \int_0^t a\,(X, s)\,\mathrm{d}s + \int_0^t b\,(X, s)\,\mathrm{d}W(s),$$

where

- $a(\cdot, \cdot)$ is drift rate, a function of X and t;
- $b(\cdot, \cdot)$ is volatility, a function of X and t.

If $X(t)$ is a Itô process, and $G(X,t)$ is a given function of $X(t)$ and t, then, $G(X,t)$ satisfies the SDE

$$\mathrm{d}G = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)\mathrm{d}t + \frac{\partial G}{\partial X}b\mathrm{d}W(t),$$

which is the so-called *Itô's lemma*.

- 2-D Taylor expansion for the given function $G\left(X,t\right)$ is

$$\Delta G=\frac{\partial G}{\partial x}\Delta X+\frac{\partial G}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta X^2\ +\frac{\partial^2 G}{\partial x\partial t}\Delta X\Delta t+\frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2+\ldots,$$

  where

$$\Delta G:=G(X+\Delta X,t+\Delta t)-G(X,t),$$

  and the terms with orders greater than Δt can be neglected.

- If $X\left(t\right)$ is a deterministic process, we get

$$\Delta G=\frac{\partial G}{\partial x}\Delta X+\frac{\partial G}{\partial t}\Delta t.$$

- If $X\left(t\right)$ is a stochastic process, we get

$$\Delta G=\frac{\partial G}{\partial X}\Delta X+\frac{\partial G}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 G}{\partial X^2}(\Delta X)^2,$$

  where the last term is of order Δt.

- Substituting the time-discretized Itô process,

$$\Delta X(t) = a\Delta t + b\varepsilon\sqrt{\Delta t}, \qquad \varepsilon \sim \mathrm{N}(0,1),$$

we derive

$$\Delta G = \frac{\partial G}{\partial X}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\varepsilon^2\Delta t,$$

since $\varepsilon \sim \mathrm{N}(0,1)$,

$$\mathbb{E}[\varepsilon] = 0, \qquad \mathrm{Var}[\varepsilon] = 1 = \mathbb{E}[\varepsilon^2] - (\mathbb{E}[\varepsilon])^2,$$

hence, we have $\mathbb{E}[\varepsilon^2] = 1$, and $\mathbb{E}[\varepsilon^2\Delta t] = \Delta t$.

- Since the order of variance of $\varepsilon^2\Delta t$ is the same as Δt^2 which can be neglected, we have

$$\Delta G = \frac{\partial G}{\partial X}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\Delta t.$$

- Let $\Delta t \to \mathrm{d}t$, we derive the SDE

$$\mathrm{d}G = \frac{\partial G}{\partial X}\mathrm{d}X + \frac{\partial G}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\mathrm{d}t.$$

- Substituting

$$\mathrm{d}X(t) = a\mathrm{d}t + b\mathrm{d}W(t),$$

we derive *Itô's lemma*

$$\mathrm{d}G = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)\mathrm{d}t + \frac{\partial G}{\partial X}b\mathrm{d}W(t).$$

- We often assume the stock price $S(t)$ follows a *geometric Brownian motion* (GBM),

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

where σ is the volatility of log-price.

- Let $G = \ln S$, we obtain

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0.$$

- By Itô's lemma, we derive

$$\mathrm{d}G(t) = \mathrm{d}\big(\ln S(t)\big) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}W(t) \;\sim\; \mathrm{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t,\ \sigma^2\mathrm{d}t\right),$$

since $\mathrm{d}W(t) \sim \mathrm{N}(0, \mathrm{d}t)$.

- $S(t)$ is non-negative, which is consistent with the limited liability of a firm.

- The continuously compounded rate of stock returns follows a normal distribution:
  - The continuously compounded rate of stock returns on the time period $[t, T]$ is
  $$\eta := \frac{\ln S(T) - \ln S(t)}{T - t},$$
  since $S(T)/S(t) = e^{\eta(T-t)}$.
  - η follows a normal distribution, i.e.,
  $$\eta \sim \mathrm{N}\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T - t}\right).$$

- Empirical evidence shows that, the continuously compounded rate of stock returns approximately follows a normal distribution under a normal market condition.

References

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3):307–327.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50(4):987–1007.