Lecture: Financial Modelling

– Option

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Outline

- **Call Option:** an option which gives the right (not obligation) to buy a certain asset[1] S for a certain price (*strike* or *exercise price*) K by a certain date (*maturity* date) T. It could be one of a virtual infinity types.

- **Put Option:** an option which gives the right (not obligation) to sell a certain asset for a certain price (*strike* or *exercise price*) by a certain date (*maturity* date).

- Options are traded on exchanges and OTC markets.

- World largest exchange for trading stock options: Chicago Board Options Exchange (CBOE).

[1] This could be one of assets (e.g. stock shares, bonds), which are called *primitives*, however, there are only a finite number of them.

Chinese options

- SSE 50 ETF option
  - traded at Shanghai Stock Exchange (SSE)
  - effective from Feb 9 2015
  - European option
- Soybean meal option
  - traded at Dalian Commodity Exchange (DCE)
  - effective from Mar 31 2017
  - American option
- Sugar option
  - traded at Zhengzhou Commodity Exchange (ZCE)
  - effective from Apr 19 2017
  - American option

- Copper option
    - traded at Shanghai Futures Exchange (SFE)
    - effective from May 21 2018
    - European option
- Cotton option
    - traded at Zhengzhou Commodity Exchange (ZCE)
    - effective from Jan 28 2019
    - American option
- Rubber option
    - traded at Shanghai Futures Exchange (SFE)
    - effective from Jan 28 2019
    - American option
- Corn option
    - traded at Dalian Commodity Exchange (DCE)
    - effective from Jan 28 2019
    - American option

- **European option** can be exercised only at maturity.
- **American option** can be exercised at any time during its life.
- **Out-of-the-money (OTM) option:** a call option where the strike price is above the price of the underlying asset; or, a put option where the strike price is below this price.
- **In-the-money (ITM) option:** a call option where the strike price is below the price of the underlying asset; or, a put option where the strike price is above this price.
- **At-the-money (ATM) option:** an option where the strike price is close to the price of the underlying asset.

- **European call**: the payoff (promised gross payment) at maturity is a function of underlying stock price at maturity, according to

$$\text{payoff}(T) = (S_T - K)^+ := \begin{cases} S_T - K, & \text{if } S_T > K, \\ 0, & \text{if } S_T \leq K, \end{cases} \tag{1}$$

where T is maturity time, $\{S_t\}_{0 \leq t \leq T}$ is stock price, K is strick price.

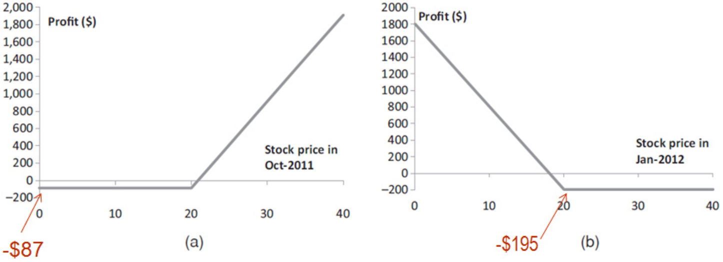
- **European put**: the payoff at maturity is a function of underlying stock price at maturity, according to

$$\text{payoff}(T) = (K - S_T)^+ := \begin{cases} K - S_T, & \text{if } K > S_T, \\ 0, & \text{if } K \leq S_T. \end{cases} \tag{2}$$

- Note that, in exchange-traded equity-option market, one option contract is usually an agreement to buy or sell 100 units of underlying shares.

Prices of Options on Intel, August 25, 2011 (stock price = $19.58)

Strike Price ($)	Calls Sept. 11	Calls Oct. 11	Calls Jan. 12	Puts Sept. 11	Puts Oct. 11	Puts Jan. 12
18	1.70	2.14	n.a.	0.09	0.57	n.a.
19	0.82	1.44	1.87	0.24	0.86	1.48
20	0.25	0.87	1.34	0.68	1.29	1.95
21	0.04	0.48	0.92	1.46	1.89	2.53



Net Profit from Long Position in an Option Contract on Intel Assuming No Early Exercise. (a) October Call with a Strike Price of $20 and (b) January Put with a Strike Price of $20.00

- Due to no arbitrage opportunity, *put-call parity* (Stoll, 1969) defines a relationship between prices of a European call, a European put of identical underlying non-dividend paying stock S, strike price K and maturity T by

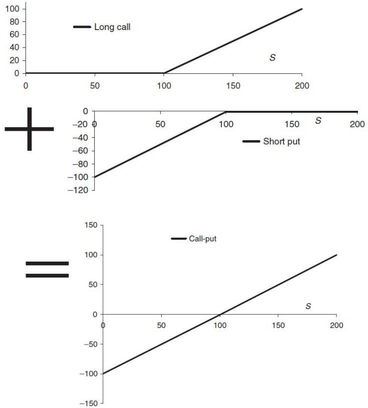
$$C(t) - P(t) \equiv S_t - e^{-r(T-t)}K, \quad \forall\, 0 \leq t \leq T, \tag{3}$$

where $C(t)$, $P(t)$ are prices of European call and put at time t, respectively.

- Imagine that you buy one European call option and write a European put option with the same strike and maturity. The payoff for this portfolio of options is

$$\max\left(S_T - K, 0\right) - \max\left(K - S_T, 0\right) = S_T - K$$

where S_T is the value of the underlying asset at time T.



- Consider the following two portfolios

  Portfolio A: one European call option plus a zero-coupon bond that provides a payoff of K at time T

  Portfolio B: one European put option plus one share of the stock.

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio B	Put option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

- If $S_T > K$, both portfolios are worth S_T at time T; if $S_T < K$, both portfolios are worth K at time T. In other words, both are worth

$$\max(S_T, K,)$$

- *Put-call parity* is model-free, no assumption for the evolution of underlying price process S_t. Note that, it does not apply to American options.

- Hence, we focus on pricing European call options.

Figure: S&P 500 Index Level (1950-2012)

Early evolution of techniques of financial modelling:

- Jules Regnault first suggested a modern theory of stock price changes used random walk in his book *Calculation of Chances and Philosophy of the Stock Exchange* (1863).
- Louis Bachelier first used the stochastic process now named Brownian motion in his PhD thesis *The Theory of Speculation* (1900).

- The classical **Black-Scholes (BS) model**[2] (Black and Scholes, 1973; Merton, 1973) assumes that, stock price $\{S_t\}_{t\geq 0}$ follows *geometric Brownian motion* (GBM)[3] under risk-neutral probability measure $\mathbb{Q}$, i.e. *stochastic differential equation* (SDE)
$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\mathrm{d}W_t, \tag{4}$$
where
  - $r > 0$ is constant default-free *instantaneous interest rate*;
  - $\sigma > 0$ is the assumed constant *instantaneous volatility* of log-return of stock price;
  - W_t is Brownian motion.
- It is a stochastic and continuous-path model.

[2] Or Black-Scholes-Merton model. BS model is the continuous-time limit of the classical discrete-time *binomial tree* model of Cox et al. (1979).

[3] Osborne (1959) first advocated GBM as a model for asset prices.

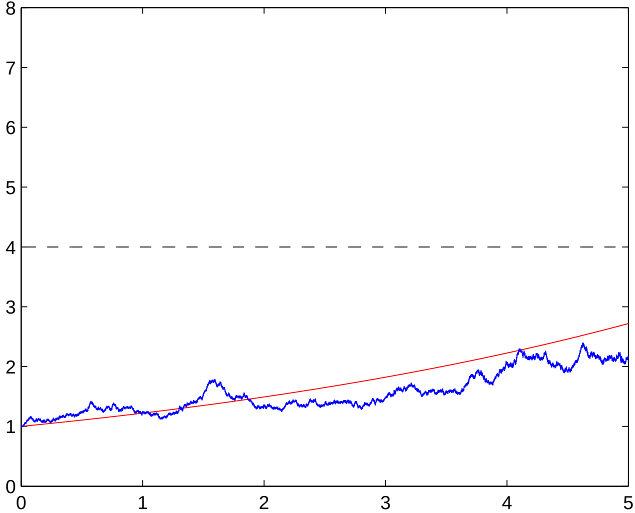


Figure: Realizations of geometric Brownian motions

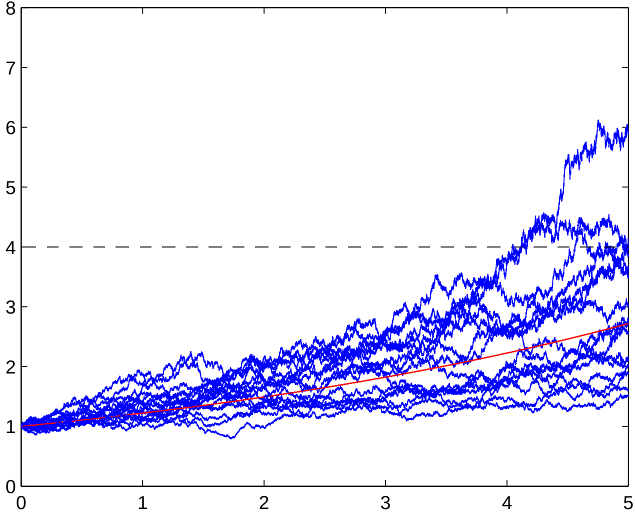


Figure: Realizations of geometric Brownian motions

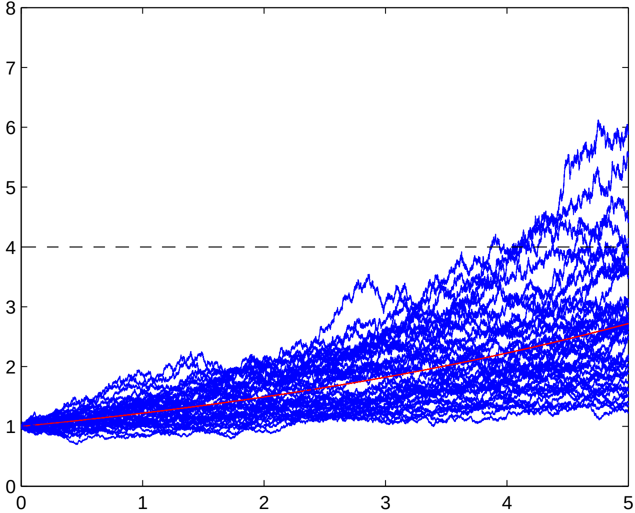


Figure: Realizations of geometric Brownian motions

- **Black-Scholes formula** for valuing a European call option at time t on a non-dividend paying stock is

$$\begin{aligned} C_{B-S}(S,K,r,\sigma;t,T) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T-K)^+ \mid \mathcal{F}_t\right] && (5)\\ &= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), && (6)\end{aligned}$$

  where $S := S_t$,

$$d_+ := \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_- := \frac{\ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

  $d_- = d_+ - \sigma\sqrt{(T-t)}$; function $\Phi(\cdot)$ is CDF of $\mathcal{N}(0,1)$, i.e.

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,\mathrm{d}u.$$

- The values of S, r, K, t, T are all observable, only volatility σ need estimation/calibration, but how?

- Note that, volatility σ is assumed for the future time period of $[t, T]$.

- **B-S implied volatility** is the volatility of underlying which when substituted into B-S formula gives a theoretical price equal to the observed market price, i.e.

$$C_{B-S}(\sigma) = \textit{Market Price}. \tag{9}$$

  - In a sense, it is market's (forward-looking and subjective) view of volatility of the underlying over the life of option.
  - In B-S model, price of European call is a strictly increasing function of σ, so, implied volatility[4] uniquely exists, i.e.

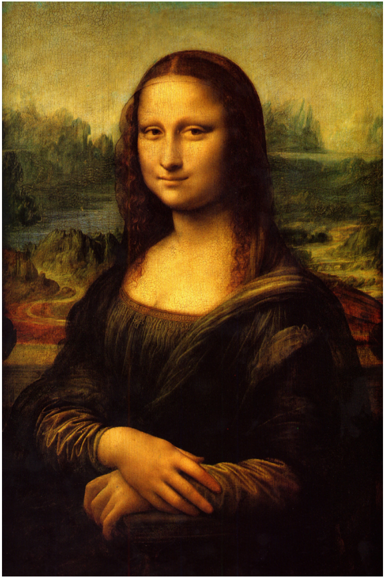
$$\sigma = C_{B-S}^{-1}(\textit{Market Price}); \tag{10}$$

  however, no analytic form exists, and numerical inverting[5] is needed.

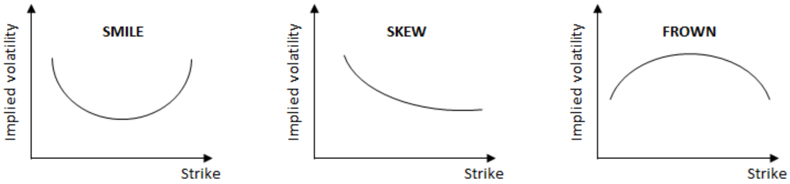
[4] B-S implied volatility is a *wrong number which, plugged into the wrong formula, gives the right answer* (Rebonato, 2005), however, option prices are commonly quoted in term of it.

[5] e.g. Newton-Raphson search (Manaster and Koehler, 1982).

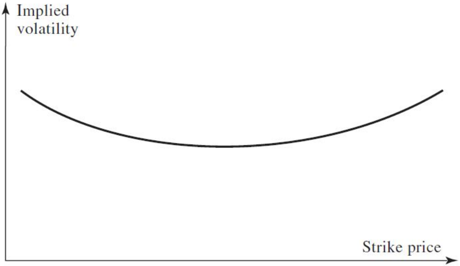
Mona Lisa's Smile :-)



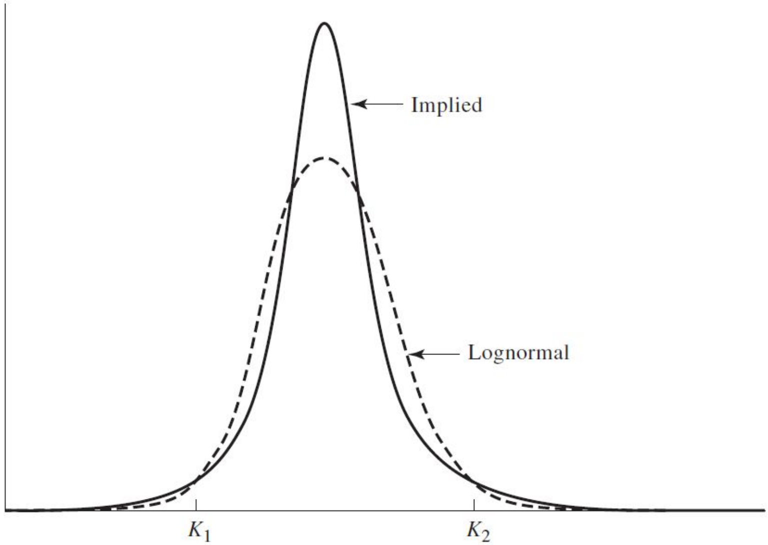
- B-S implied volatilities often present "*volatility smile*" for options of different strikes on identical underlying (Rubinstein, 1985):



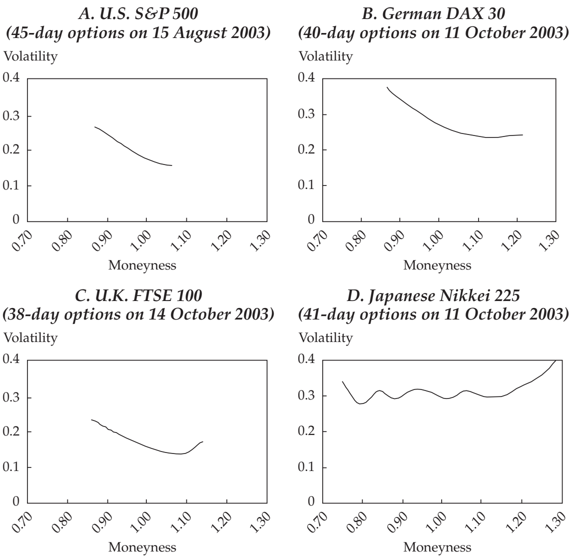
- The volatility smile used by traders to price foreign currency options has the general form shown in figure below. The implied volatility is relatively low for at-the-money options. It becomes progressively higher as an option moves either into the money or out of the money.



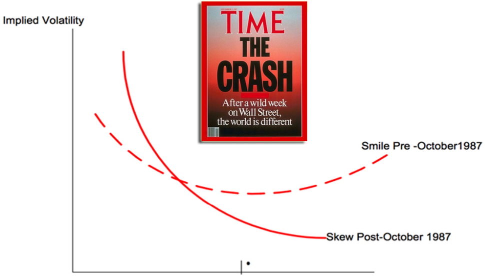
Implied and lognormal distribution for foreign currency options



B-S Implied Volatility: Empirical Volatility Smiles on Stock Indexes (Jackwerth, 2004)



Note: Moneyness = Strike price/Index level.

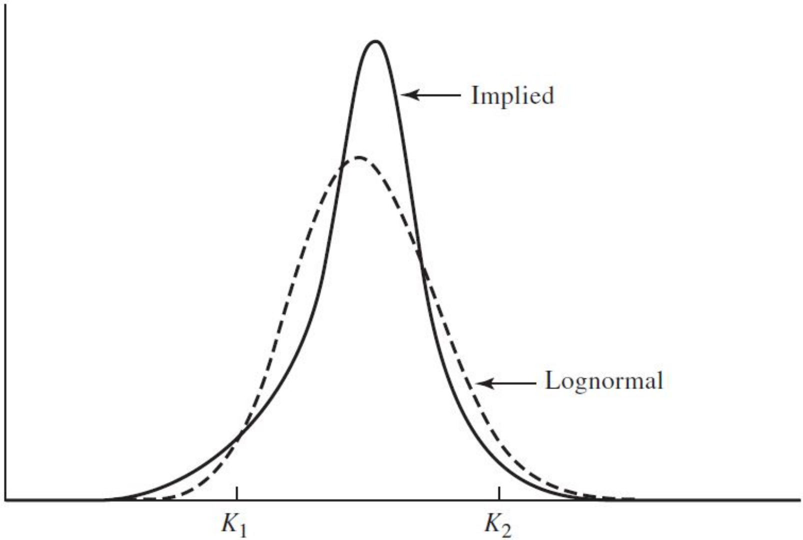


- **Skew** or **smirk**: as found by Bates (1991), after *1987's Wall Street crash* or *Black Monday* (-22.9% return on SPX on October 19, 1987). OTM calls tend to be less expensive than suggested by the options models, while OTM puts[6] tend to be more expensive, see Bollen and Whaley (2004); Garleanu et al. (2009); Bondarenko (2014).

- New pricing models are needed!

- Crash of '87 also gave birth to *high-frequency trading* (HFT): Nasdaq made mandatory for market makers to instantly execute trades of 1,000 shares or less by retail investors in its *Small Order Execution System* (SOES). Market makers and institutional investors, who usually traded in much bigger chunks of stock, had to wait in line behind SOES traders.

[6]OTM puts essentially provide portfolio insurance, as they pay off when the market crashes.


Implied
Lognormal
K_1
K_2

- **Statistical volatility** (or, **realized volatility**) is the actual volatility directly estimated from time series of historical prices of underlying stock.
  - In a sense, it is historical summary of volatility of underlying.
  - In B-S model, stock price is assumed to follow *geometric Brownian motion*, then, statistical volatility is standard deviation of time series of continuously-compounded log-returns $\{R_1, R_2, ..., R_T\}$ where

$$R_t := \ln S_t - \ln S_{t-1}, \qquad t = 1, 2, ..., T-1. \tag{11}$$

  - For high-frequency intraday data, the formula of *realized volatility* is different, see Andersen et al. (2001), Andersen et al. (2003).

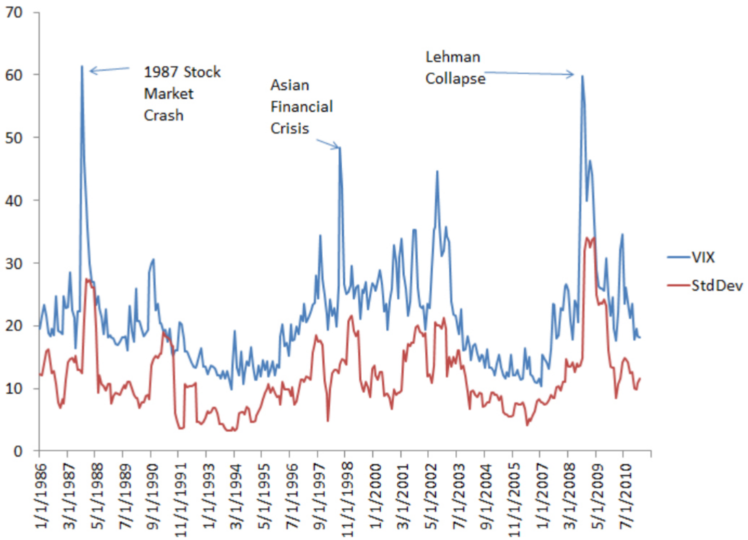


Figure: Monthly *implied volatility* (VIX) v.s. *statistical volatility* of S&P 500, *01/1986 – 03/2011*

- B-S implied volatility is widely believed to be a good estimate of "market's" expectation of the volatility of underlying asset over the remaining life of the option. If option markets are efficient, implied volatility should be an efficient forecast of future volatility, i.e., implied volatility should subsume the information contained in all other variables in the market information set in explaining future volatility (Christensen and Prabhala, 1998).

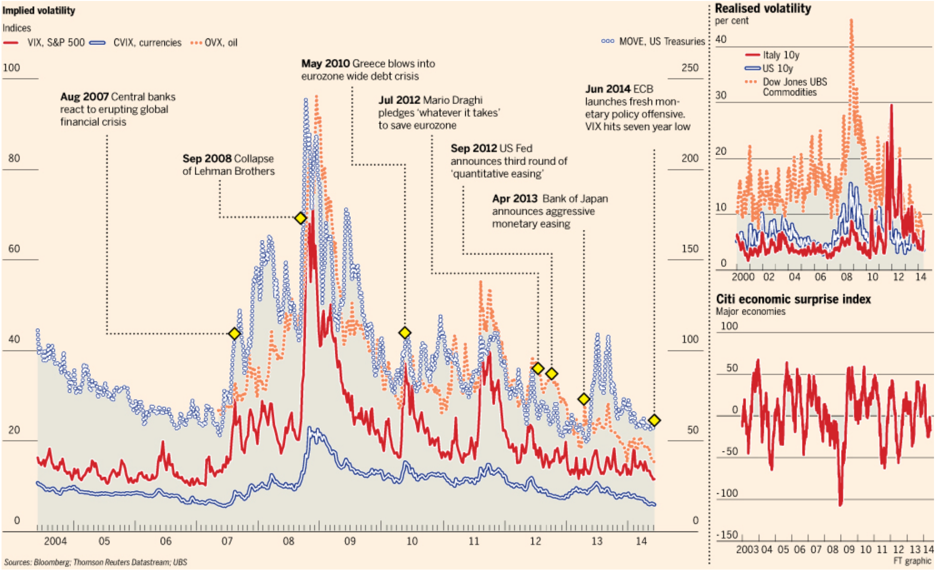
- Researchers often use B-S implied volatility in other models as an ex ante measure of perceived asset price risk (Poterba and Summers, 1986; Stein, 1989).

- Empirical studies on the relationship between the two volatilities: Latane and Rendleman (1976), Chiras and Manaster (1978), Beckers (1981), Day and Lewis (1992), Canina and Figlewski (1993), Lamoureux and Lastrapes (1993), Jorion (1995), Christensen and Prabhala (1998), Blair et al. (2001); theories in Bergman et al. (1996).

- Note that,

$$\textit{Risk-neutral Probability} = \textit{Subjective Probability} \times \textit{Risk-aversion Adjustment}.$$

Volatility in Option Pricing Model: Fear Gauge VIX



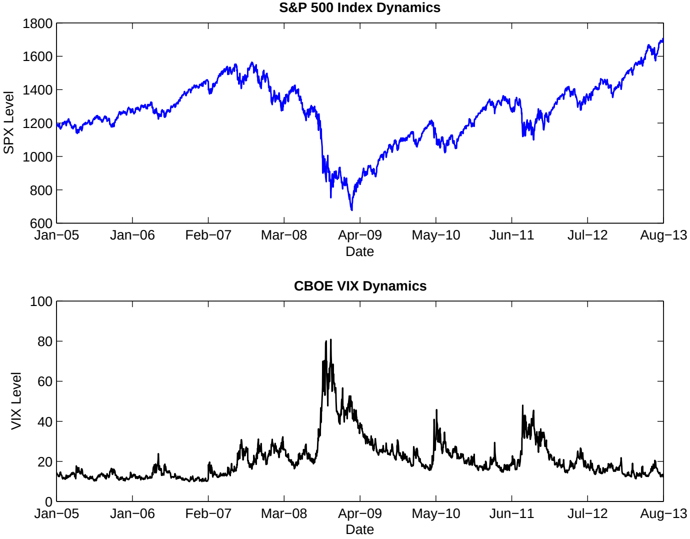


Figure: VIX v.s. SPX (Standard & Poor's 500 Index), *03/01/2005 – 02/08/2013*

- VIX tends to rise when the S&P 500 falls; VIX tends to decline or remain constant when the S&P 500 rises.

- Pre-crisis-, crisis-, and post-crisis correlation coefficients for the SPX and VIX:

Correlation Coefficients between SPX– and VIX Levels		
03/01/2005 – 02/08/2013 -0.6539	05/07/2007 – 02/08/2013 -0.7261	20/07/2009 – 02/08/2013 -0.6670
03/01/2005 – 02/05/2008 0.3259	05/05/2008 – 01/07/2010 -0.6905	02/07/2010 – 02/08/2013 -0.6971

- **Front Office** for trading function
  - hedges risks by ensuring that exposures to individual market variables are not too great (e.g. Greek letters).
- **Middle Office** concerned with the overall level of the risks being taken, capital adequacy and regulatory compliance
  - aggregates the exposures of all traders to determine whether the total risk is acceptable (e.g. Value at Risk).
- **Back Office** for record/archives keeping function

- **Greek Letters**: Delta (Δ), Gamma (Γ), Vega (ν), Theta (Θ), Rho (ρ), ...
- Each measures the **sensitivity** from a different aspect of the risk in a trading position.
- Traders calculate their Greeks at the end of each day, and are required to take action if the internal risk limits are exceeded.
- Failure to take this action is liable to lead to immediate dismissal.

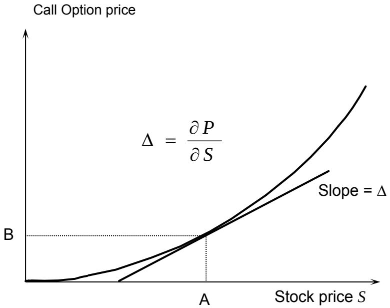
- **Delta** of an asset (or an asset portfolio) is partial derivative of a asset (or an asset portfolio) price P with respect to price of underlying asset, i.e.

$$\Delta := \frac{\partial P}{\partial S} \approx \frac{P(S+\Delta S) - P(S)}{\Delta S}, \tag{12}$$

where ΔS is a small increment, P is asset (or a portfolio) value.

  - In B-S model, for a European call on a non-dividend stock,

$$\Delta = \frac{\partial C_{B-S}(S, K, r, \sigma; t, T)}{\partial S} = \Phi(d_+). \tag{13}$$



Example (Static Delta-neutral Hedging for An Asset Portfolio)

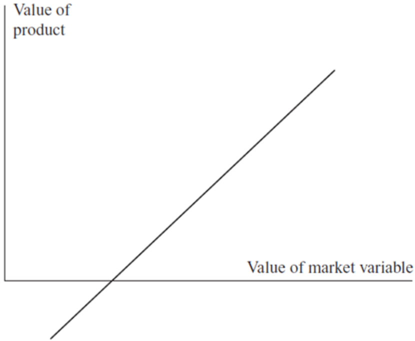
- Suppose that, a $\$0.1$ increase in price of stock leads to asset portfolio decreasing in value by $\$100$, then, delta of this asset is $-100/0.1 = -1000$.
- **Delta hedging**: this asset portfolio could be hedged against short-term changes in price of stock by buying 1000 unites of stock.
- **Portfolio delta neutral**: $P(S) + 1000 \times S$, in general,
$$P(S) - \Delta_S \times S.$$
- Assumption: delta hedging does not influence stock price.

- **Linear Product**: its value at any given time is linearly dependent on the value of underlying market variable.

- e.g. forward contracts

- both small and large movements of underlying values can be well hedged

- a simple (static) hedge & forget strategy (i.e. delta neutral $P(S) - \Delta \times S$) can be used: once it has been set up, never needs to be changed

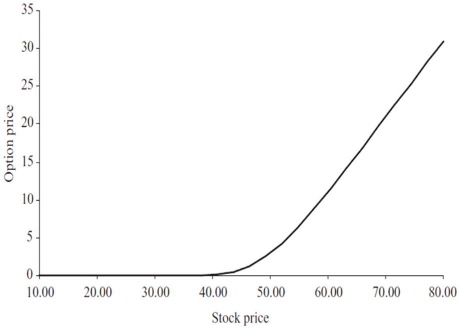


- **Nonlinear product**: relationship between the value of product and the underlying asset price at any given time is nonlinear.

- e.g. options.

- Delta neutral only protects against small movements in the price of the underlying asset.

- It requires the hedge to be consistently/frequently rebalanced to preserve delta neutrality (i.e. dynamic hedging).



Example (Dynamic Delta-neutral Hedging for A Short Call in Discrete Time)

- A trader at a bank has sold (shorted) for \$300,000 a European call on 100,000 shares of a non-dividend paying stock.
- Parameter setting: $t = 0$, $S_0 = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$, $T = 20$ weeks.
- Theoretical value of this option under Black-Scholes model is \$240,000.
- So, option has been sold \$60,000 more than theoretical value.
- How does the bank hedge its risk to lock in a \$60,000 profit?

- There are 100,000 units of options.
- B-S value of one option is \$2.40.
- Initially, delta per option is 0.522.
- Delta of the position is 52,200.
- This means that 52,200 unites of shares must purchased to create a delta neutral position.
- But, if a week later, stock price falls by the end of the first week to \$48.12, delta falls to 0.458, then, $52,200 - 45,800 = 6,400$ unites of shares must be sold to maintain delta neutrality.
- To preserve delta neutrality, hedge has to be adjusted periodically (rebalanced weekly).
- Tables 7.2, 7.3 provide examples of how weekly delta hedging might work for this option.

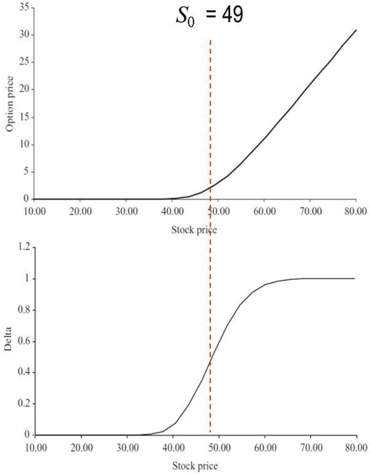


TABLE 7.2 Simulation of Delta Hedging (Option closes in-the-money and cost of hedging is $263,300.)

Week	Stock Price	Delta	Shares Purchased	Cost of Shares Purchased ($000)	Cumulative Cash Outflow ($000)	Interest Cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

Simulated path

Option exercised

2557.8
-308.0
+2.5
=$2252.3k

Cost
=5263.3-100*50
= $263.3k
vs B-S'=$240k

TABLE 7.3 Simulation of Delta Hedging (Option closes out-of-the-money and cost of hedging = $256,600)

Week	Stock Price	Delta	Shares Purchased	Cost of Shares Purchased ($000)	Cumulative Cash Outflow ($000)	Interest Cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

Simulated path

Option not exercised

Cost = $256.6k
vs B-S'=$240k

- Under BS model with no transaction cost, if the *dynamic hedging* scheme (Derman and Taleb, 2005) using underlying asset worked perfectly in continuous time, the cost of hedging would, after discounting, be exactly equal to BS price (here $\$240k$) for every simulated path of stock price.
- The reason for the variation in the cost of delta hedging is that, it is rebalanced only once a week.
- As rebalancing takes place more frequently, the variation is reduced.

Where does the cost of hedging come from?

- The delta-hedging procedure in Tables 7.2, 7.3 in effect synthetically creates a long option position to neutralize the trader's short option position.
- This hedging strategy tends to involve selling stock just after the price has gone down and buying stock just after the price has gone up – a buy-high sell-low scheme.
- The theoretical cost of hedging $\$240k$ comes from the average difference between the price paid for the stock and the price realized for it and cost of borrowing.

- Gamma (Γ) is the rate of change of delta (Δ) with respect to underlying price, i.e.

$$\Gamma := \frac{\partial \Delta}{\partial S} = \frac{\partial^2 P}{\partial S^2}. \tag{14}$$

  - In B-S model, for European call option on a non-dividend stock,

$$\Gamma = \frac{\partial^2 C_{B-S}(S, K, r, \sigma; t, T)}{\partial S^2} = \frac{\Phi'(d_+)}{\sigma S \sqrt{T-t}}. \tag{15}$$

  - If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.
  - If gamma is large, delta is highly sensitive to the price of underlying.

- It is very risky to leave a delta-neutral portfolio (of nonlinear with big Gamma) unchanged for any length of time.

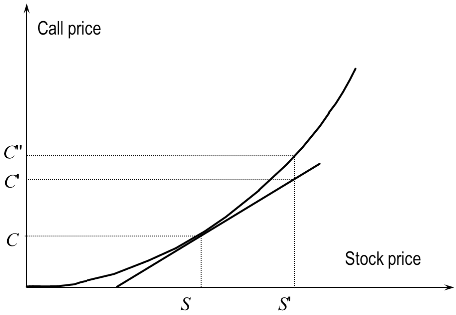
- When stock price moves from S to S', delta hedging assumes that option price moves from C to C', when in fact it moves from C to C''.

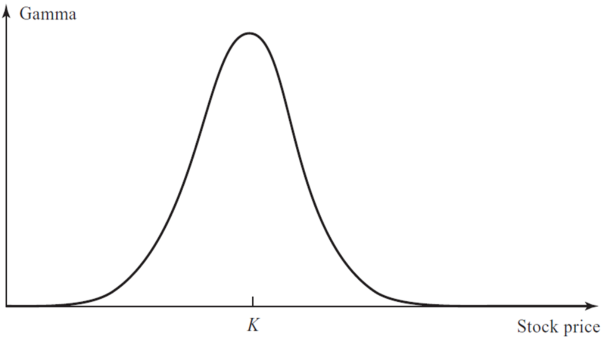
- The difference between C' and C'' leads to a hedging error.

- This error depends on the curvature of relationship between option price and underlying price.

- Gamma measures this curvature.

- Gamma is greatest for options where the stock price is close to strike price.





Relationship between Gamma of an Option and Price of Underlying Stock where K is the Option's Strike Price

- How does it change when approaching to maturity?

- Vega (ν) is the rate of change of the value of of a derivative (or portfolio of derivatives) with respect to volatility, i.e.

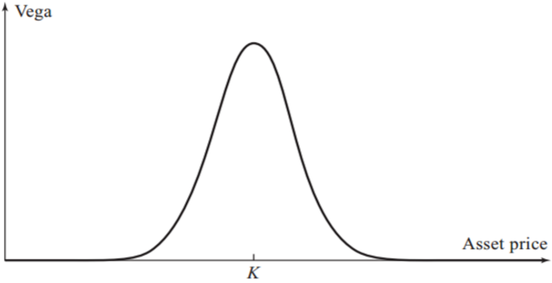
$$\nu := \frac{\partial P}{\partial \sigma}. \tag{16}$$

  - In B-S model, for a European call option on a non-dividend stock,

$$\nu = \frac{\partial C_{B-S}(S, K, r, \sigma; t, T)}{\partial \sigma} = S\Phi'(d_+)\sqrt{T-t}. \tag{17}$$

- This is different from the other Greeks, since it is a derivative with respect to a parameter rather than a variable.
- The volatility of a market variable measures our uncertainty about the future value of the variable.
- In option valuation models, volatilities are often assumed to be constant; in practice, volatilities change through time.
- Spot positions and forwards do not depend on the volatility of asset prices; but options and more complicated derivatives (e.g. variance swap) do.

- Like Gamma, Vega tends to be greatest for options at the strike price K.



Variation of Vega of an Option with Price of Underlying Asset where K is Option's Strike Price

- Δ can be changed by taking a position in the underlying.
- To adjust Γ & ν, it is necessary to additionally take a position in an option or other derivatives.

Example (Delta, Gamma & Vega Neutral)

- Consider one delta-neutral portfolio and two types of options:

	Delta	Gamma	Vega
Portfolio	0	$-5{,}000$	$-8{,}000$
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

- Gamma & Vega neutralisation: buy w_1 units of Option 1 and w_2 units of Option 2, then,

$$-5000 + 0.5w_1 + 0.8w_2 = 0,$$
$$-8000 + 2.0w_1 + 1.2w_2 = 0,$$

  with the solution $w_1 = 400$, $w_2 = 6000$.
- Delta neutralisation: sell $400 \times 0.6 + 6,000 \times 0.5 = 3,240$ units of underlying assets.

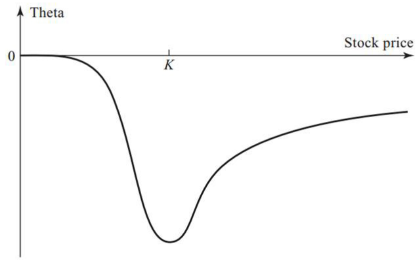
- Theta (Θ) (time decay) of a derivative (or portfolio of derivatives) is the rate of change of value with respect to passage of time, i.e.

$$\Theta := \frac{\partial P}{\partial t}. \tag{18}$$

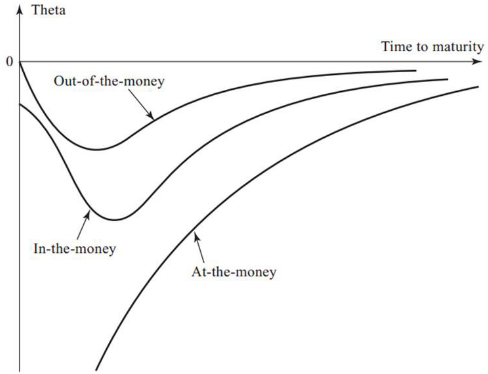
  - In B-S model, for a European call option on a non-dividend stock,

$$\Theta = \frac{\partial C_{B-S}(S, K, r, \sigma; t, T)}{\partial t} = -\frac{\sigma S \Phi'(d_+)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_-).$$

- Theta of a call or put is usually negative.
  - If time passes with the price of the underlying asset and all other variables remaining the same, the value of the option declines.
- Theta is not the same type of Greek letter as delta.
  - There is uncertainty about a future asset price, but there is no uncertainty about the passage of time.
  - It makes sense to hedge against changes in the price of an underlying asset, but it does not make any sense to hedge against the effect of the passage of time on an option portfolio.



Variation of Theta of a European Call Option with Stock Price where K is Option's Strike Price



Typical Patterns for Variation of Theta of a European Call Option with Time to Maturity

- Rho (ρ) is the partial derivative with respect to a parallel shift in all interest rates, i.e.
$$\rho := \frac{\partial P}{\partial r}. \tag{19}$$
  - In B-S model, for a European call option on a non-dividend stock,
$$\rho = \frac{\partial C_{B-S}(S, K, r, \sigma; t, T)}{\partial r} = K(T-t)e^{-r(T-t)}\Phi(d_-). \tag{20}$$

How the change in portfolio value in a short time period depending on Greek letters?

$$\Delta P = \frac{\partial P}{\partial S}\Delta S + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 P}{\partial S \partial t}(\Delta S \Delta t) + \ldots, \tag{21}$$

where

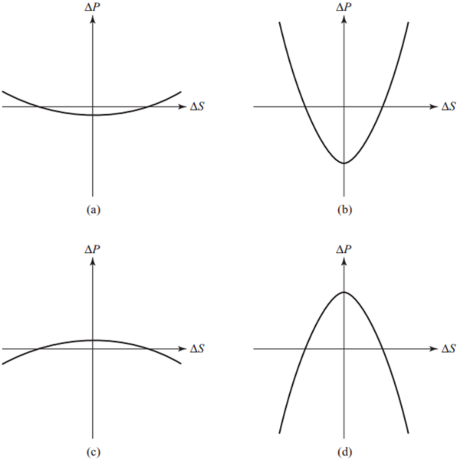
$$\Delta P := P(S + \Delta S, t + \Delta t) - P(S, t), \tag{22}$$

- the volatility of underlying asset and interest rates are assumed to be constant;
- terms of higher order than Δt are ignored.

- For a delta-neutral portfolio,

$$\Delta P \approx \Theta \Delta t + \frac{1}{2}\Gamma(\Delta S)^2. \tag{23}$$

Parabolic Curve



(a) slightly positive gamma
(b) large positive gamma

(c) slightly negative gamma
(d) large negative gamma

- When the volatility of underlying asset is uncertain,

$$\Delta P = \frac{\partial P}{\partial S}\Delta S + \frac{\partial P}{\partial \sigma}\Delta\sigma + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial \sigma^2}(\Delta\sigma)^2 + \dots, \tag{24}$$

where

$$\Delta P := P(S + \Delta S, \sigma + \Delta\sigma, t + \Delta t) - P(S, \sigma, t), \tag{25}$$

- 1^{st} term is eliminated by delta hedging;
- 2^{nd} term is eliminated by making the portfolio vega neutral;
- 3^{rd} term is deterministic (non-stochastic);
- 4^{th} term is eliminated by making the portfolio gamma neutral.
- 5^{th} traders often define other "Greek letters" to correspond to higher order terms in Taylor Series Expansion. For example, $\frac{\partial^2 P}{\partial \sigma^2}$ is sometimes referred to as "gamma of vega".

- In B-S model, the price of a single derivative dependent on a non-dividend-paying stock satisfies the B-S PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{26}$$

- Then, the value of a portfolio of such derivatives also satisfies the PDE

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = rP, \tag{27}$$

$$\Theta := \frac{\partial P}{\partial t}, \quad \Delta := \frac{\partial P}{\partial S}, \quad \Gamma := \frac{\partial^2 P}{\partial S^2}, \tag{28}$$

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rP. \tag{29}$$

- For a delta-neutral portfolio

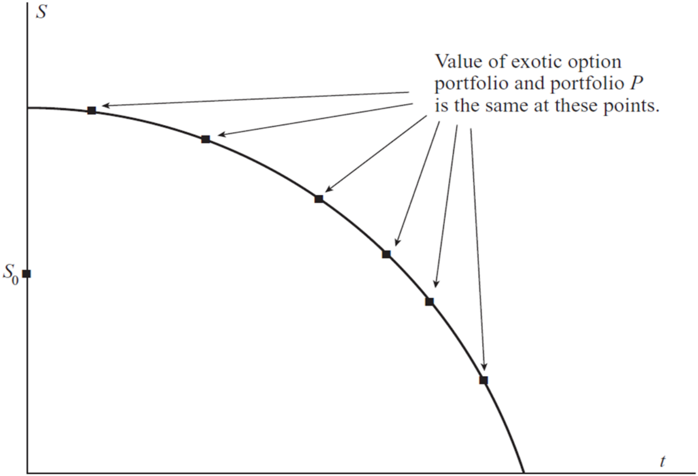
$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rP. \tag{30}$$

- Traders usually ensure that their portfolios are delta-neutral at least once a day.
    - Whenever opportunity arises (by finding proper options or other nonlinear derivatives that can be traded in the volume required at competitive prices), they improve gamma and vega.
- Maintaining delta neutrality for an individual option on an asset by trading the asset daily would be very expensive.
- As portfolio becomes larger, hedging as a whole becomes less expensive (by large economies of scale).
- Cost of daily rebalancing is covered by the profit on many different trades.

- **Dynamic hedging**, in theory, requires hedging over the next infinitesimal time interval; in practice, needs frequent trading and rebalancing, hence may incur significant transaction costs.

- Dynamic hedging in incomplete markets: *minimum-variance criterion* over hedging error (Basak and Chabakauri, 2012).

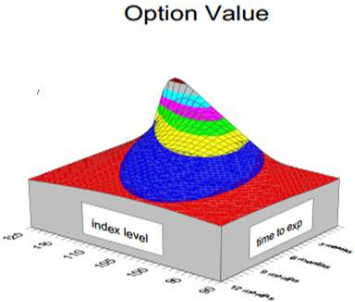
- **Static hedging** (Derman et al., 1995; Carr et al., 1998) is a one-time fixed strategy created to hedge an existing option or position.
  - Once created, it is not adjusted at all, contrary to a dynamic hedge.
- **Static options replication** involves approximately replicating an exotic option with a static portfolio of vanilla options (e.g. standard options with different strikes and maturities but fixed portfolio weights).
  - Once constructed, this portfolio will replicate the value of target option for a wide range of stock prices and times before expiration, without requiring further weight adjustments (i.e. static replication).
  - The exotic option is hedged by shorting this portfolio.
  - Static options replication is contrasted with dynamic options replication where we have to trade continuously to match the option.
  - It minimizes dynamic hedging risk and costs.

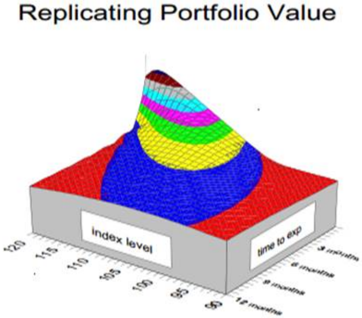


Static Options Replication

A replicating portfolio, P is chosen so that it has the same value as the exotic option portfolio at a number of points on a boundary.

Example: Static Hedging An Exotic Option with A Portfolio of Standard Options





- The graph on the left shows the value of a one-year up-and-out call, struck at 100 with out-barrier at 120, for all times to expiration and for market levels between 90 and 120.
- The graph on the right shows the value of a replicating portfolio constructed from seven standard options, struck either at 100 or 120, and expiring every two months over the one-year period.
- The replicating portfolio value approximately matches the target option value over a large range of times and stock prices, and has the same general behavior.
- The more standard options you include in the replicating portfolio, the better the match.

- If we assume that, markets exist trading for European options of **all** (i.e. continuum from 0 to ∞) strikes, then, any smooth payoff function (i.e. twice continuously differentiable) at maturity time T, $f(S_T)$, with final future price S_T, can be spanned as (Carr and Madan, 2001b)

$$\begin{aligned} f(S_T) &= f(\kappa) + f'(\kappa)\left[\left(S_T-\kappa\right)^+ - \left(\kappa - S_T\right)^+\right] \\ &\quad + \int_0^\kappa \left(K - S_T\right)^+ f''(K)\mathrm{d}K + \int_\kappa^\infty \left(S_T - K\right)^+ f''(K)\mathrm{d}K, \qquad \forall \kappa \geq 0, \end{aligned}$$

  where
  1. 1^{th} is payoff from a static position in $f(\kappa)$ pure discount bonds, each paying \$1 at T;
  2. 2^{nd} is payoff from long $f'(\kappa)$ calls struck at κ and short $f'(\kappa)$ puts also struck at κ;
  3. 3^{rd} is payoff from a static position in long $f''(K)\mathrm{d}K$ puts at all strikes less than κ;
  4. 4^{th} is payoff from a static position in $f''(K)\mathrm{d}K$ calls at all strikes greater than κ.

- Note that, $\left(S_T-\kappa\right)^+ - \left(\kappa - S_T\right)^+ = S_T - \kappa$.

- In absence of arbitrage and frictionless market, i.e. applying risk-neutral valuation to both sides, total cost of static hedging at time 0 for payoff f at time T is

$$\begin{aligned} V_0^f(T) &= \mathbb{E}^{\mathbb{Q}}\left[B_0(T)f(S_T)\right] = B_0(T)f(\kappa) + f'(\kappa)\left[C_0(\kappa, T) - P_0(\kappa, T)\right] \\ &\quad + \int_0^{\kappa} P_0(K, T)f''(K)\mathrm{d}K + \int_{\kappa}^{\infty} C_0(K, T)f''(K)\mathrm{d}K, \end{aligned}$$

  where
  - $B_0(T)$ denote the present value at time 0 of risk-free bond paying $\$1$ at maturity T;
  - $P_0(K, T)$, $C_0(K, T)$ denote initial prices at time 0 of put and call struck at K with maturity T, respectively.

- No assumption was made regarding to the stochastic process governing the underlying price!

- This observation was first noted in Breeden and Litzenberger (1978) and Banz and Miller (1978), and then established formally in Green and Jarrow (1987), Nachman (1988), Bakshi and Madan (2000), Carr and Madan (2001a,b), Bakshi et al. (2003).

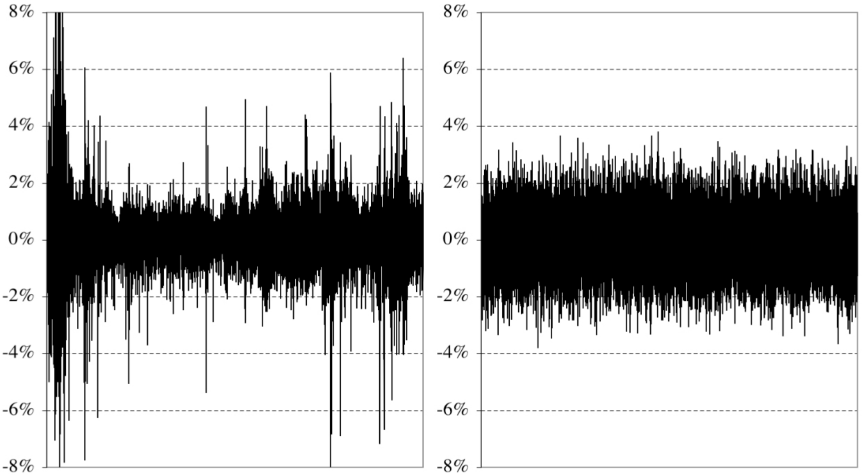


Figure: Daily returns of the DJIA 01/1930 - 08/2006 **(Left)** v.s. 76.5 years of Brownian motion with identical volatility **(Right)**

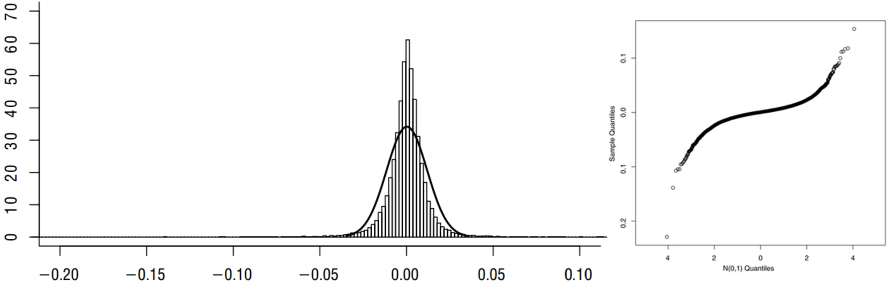


Figure: Frequency distribution of (77 years from 1928 to 2005 of) SPX daily log returns (-22.9% return on October 19, 1987) compared with the normal distribution **(Left)**, and Q-Q plot of SPX daily log returns compared with the normal distribution **(Right)**

- Statistic test for normal distribution: e.g. *Jarque-Bera (JB) test* (Jarque and Bera, 1987).

- From many empirical investigations (Kendall and Hill, 1953; Mandelbrot, 1963; Cont, 2001), we find that,
  1. **Return Distribution**: *asymmetric leptokurtic* features, i.e. return distribution is skewed to left (i.e. negative skewness), and has a higher peak and two heavier tails than those of normal distribution (Fama, 1965);
  2. **Jump Component**: market portfolio contains jumps, e.g. Jarrow and Rosenfeld (1984), Schwert (1990).
  3. **Market Price**: implied volatility 'smile' (before 1987's Wall Street crash) or 'skew' (post 1987's Wall Street crash) in option markets (Bates, 2000).

- **Volatility surface** is the surface formed by mapping *implied volatility* as a function of strike and time-to-maturity (Cont et al., 2002).

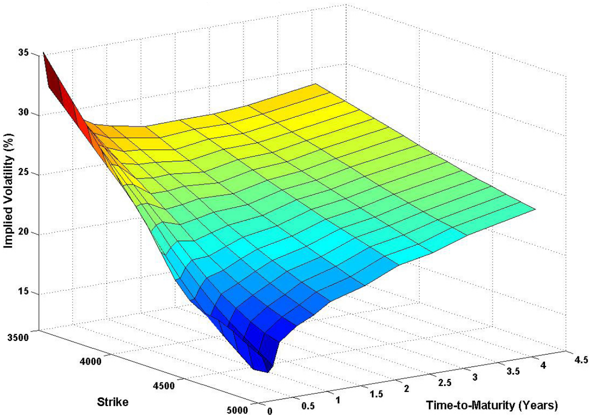


Figure: A snapshot of volatility surface for Eurostoxx 50 index on 28 Nov 2007

- These evidences show that, Black-Scholes' *geometric Brownian motion with drift* is not an accurate model for underlying asset prices in real financial market.

- The core problem: how can we develop pricing models to insure that all instruments are consistently priced with respect to each other throughout the time – that is, to satisfy the golden rule of absence of arbitrage?

- Every option pricing model basically has to make three basic assumptions:
  1. underlying price process (i.e. distributional assumption),
  2. interest rate process,
  3. market price of factor risks.

- Volatility plays a central role in option pricing.

- What are option prices, if instantaneous volatility σ in BS model becomes time-deterministic only and not stochastic (Merton, 1973), say, $\sigma(t) > 0$?

- **Local (deterministic) volatility models**[7]: instantaneous volatility is merely a deterministic function of time and underlying spot price, i.e. $\sigma(S_t, t)$, e.g. *constant elasticity of variance* (CEV) model of Cox (1975), **Dupire (1994)**, Derman and Kani (1994), Rubinstein (1985), Dumas et al. (1998).

[7] Its discrete-time version is the *implied binomial tree* model (Rubinstein, 1994) ; whereas the discrete-time version of B-S model is *binomial tree* model (Cox et al., 1979).

- **Stochastic volatility (SV) models**: instantaneous volatility has a randomness of its own, e.g. Johnson and Shanno (1987), *GBM* **(Hull and White, 1987)**, Scott (1987), Wiggins (1987), Stein and Stein (1991); *CIR* **(Heston, 1993)**[8]; GARCH of Duan (1995), Heston and Nandi (2000), Christoffersen et al. (2013) and Ornthanalai (2014); regime switching of Bollen et al. (2000),Britten-Jones and Neuberger (2000) and Duan et al. (2002); *SABR* (stochastic alpha, beta, rho) (Hagan et al., 2002), *ambiguous volatility* (Epstein and Ji, 2013).

[8] The most popular SV model for pricing equity options, see details of implementation in Rouah (2013); its discretised-time version is the *affine GARCH(1,1)* SV model (Nelson and Foster, 1994; Heston and Nandi, 2000).

- **Poisson jump-diffusion models**: non-systematic (i.e. diversifiable) jumps of **Merton (1976)**, Kou (2002), finite number of random jump sizes of Jones (1984), Ball and Torous (1985), Ahn and Thompson (1988), Back (1991), systematic jumps of Ahn (1992), Bates (1996), Scott (1997), Jorion (1988), local volatility with Poisson-jump models of Andersen and Andreasen (2000), Eraker et al. (2003).

- Standard measures for the performance of alterative option pricing models:
  - **(1)** internal consistency of implied parameters/volatility with relevant time series data;
  - **(2)** out-of-sample pricing;
  - **(3)** hedging.

  Bakshi et al. (1997) found that,
  - incorporating stochastic volatility and jumps is important for **(1)** and **(2)**,
  - however, modeling stochastic volatility alone yields the best performance for **(3)**.

1. **Monte Carlo simulation approach** (Glasserman, 2003):
   - **(1)** **discretization simulation**: Boyle (1977), Duffie and Glynn (1995), Boyle et al. (1997), Longstaff and Schwartz (2001), Rogers (2002);
   - **(2)** **exact simulation**: Broadie and Kaya (2006), Andersen (2008);
   - **(3)** **importance sampling (via change of measure)**: Asmussen and Glynn (2007).
2. **Tree (or lattice) approach**: *binomial tree* of Cox et al. (1979), *trinomial tree*.
3. **PDE approach**: Jarrow (1999), Carr and Cousot (2011).
4. **Characteristic function / Fourier transform approach**: Heston (1993), *Fast Fourier transform* (FFT) of Carr and Madan (1999), Bakshi and Madan (2000), Duffie et al. (2000), Lee (2004).
5. **Other transform methods**: *Laplace transform* of Kou et al. (2005); *Esscher transform* of Gerber and Shiu (1994).
6. **Time change approach**: Mandelbrot and Taylor (1967), Clark (1973), Monroe (1978), Ané and Geman (2000), Geman et al. (2001), Carr et al. (2003), Carr and Wu (2004), Huang and Wu (2004), Geman (2005), Li and Linetsky (2014).
7. **Expansion methods**: Spectral methods (Linetsky, 2007), Kristensen and Mele (2011), *Hermite polynomial expansion* (Xiu, 2014), *Edgeworth expansion* (Heston and Rossi, 2017).

1. *The Volatility Surface: A Practitioner's Guide* (Gatheral, 2006), by Jim Gatheral, former (MD) head of Equity Quantitative Analytics group at Merrill Lynch
2. *The Volatility Smile* (Derman and Miller, 2016)
3. *Option Valuation under Stochastic Volatility (I&II): With Mathematica Code* (Lewis, 2000, 2016)
4. *Financial Modelling with Jump Processes* (Cont and Tankov, 2004)
5. *Lévy Processes in Finance: Pricing Financial Derivatives* (Schoutens, 2003)
6. *Exotic option pricing and advanced Lévy models* (Kyprianou et al., 2006)
7. *Stochastic Volatility: Selected Readings* (Shephard, 2005)

- Exotic options are usually traded in OTC market, much less liquid, smaller trading volume than the plain vanilla derivatives in exchanges.
- However, they are important to a bank, as profit tends to be much higher than plain vanilla.
- *Barrier option* is a type of *path-dependent options* that come into existence or disappear when the price of underlying asset reaches a certain barrier (e.g. *knock-out* or *knock-in* option).
- Once the underlying process reaches those barriers, the values of those barrier options will be zero forever (*knock out*), or be effective (*knock in*) .
- Usually analytical formulas for the prices of complex barrier options don't exist in general, then, we need numerical methods.
- Why barrier options? cheaper than a standard option; provides a payoff distribution that better matches a hedger's risk or a speculator's veiw.

- **Knock-out call option** is a type of barrier option that the right to exercise the underlying *European call option* is lost (or *knocked out*) and the option becomes worthless, if the predetermined price barrier is crossed by the option's underlying stock before expiration.

- Consider a knock-out call option (or *up-and-out call – U&O call*) which will expire on T with upper barrier $B > S_0$ where $S_0 > 0$ is initial stock price. Its payoff at maturity T is

$$\text{payoff}(T) = (S_T - K)^+ \mathbb{1}\{M_T < B\} = \begin{cases} (S_T - K)^+, & \text{if } M_T < B, \\ 0, & \text{if } M_T \geq B, \end{cases} \tag{31}$$

where

$$M_T := \max_{0 \leq u \leq T} \{S_u\}. \tag{32}$$

- We assume strick price $K < B$; otherwise, the option must knock out in order to be in the money and hence could only have zero payoff.

- Assume underlying stock price $\{S_t\}_{0\leq t\leq T}$ follows a geometric Brownian motion[9], i.e.

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{33}$$

and the barrier is continuously-monitored.

- What is present value of this knock-out call option?

$$C_{U\&O}(S, K, r, \sigma; 0, T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \mathbb{1}\{M_T < B\}\right], \quad S_0 = S. \tag{34}$$

[9] Brownian motion is the scaling limit of random walk (Fama, 1995) in dimension one.

- The present value of this knock-out call option has analytic formula (Merton, 1973; Rubinstein and Reiner, 1991)[10]:

$$C_{U\&O}(S,K,r,\sigma;0,T) \quad = \quad SI_1 - KI_2 - SI_3 + KI_4,$$

where

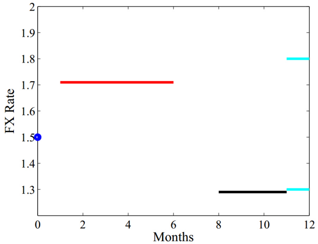
$$\begin{aligned}
I_1 &:= \Phi\left(\delta_+\left(T,\frac{S}{K}\right)\right) - \Phi\left(\delta_+\left(T,\frac{S}{B}\right)\right),\\
I_2 &:= e^{rT}\left[\Phi\left(\delta_-\left(T,\frac{S}{K}\right)\right) - \Phi\left(\delta_-\left(T,\frac{S}{B}\right)\right)\right],\\
I_3 &:= \left(\frac{S}{B}\right)^{-\frac{2r}{\sigma^2}-1}\left[\Phi\left(\delta_+\left(T,\frac{B^2}{KS}\right)\right) - \Phi\left(\delta_+\left(T,\frac{B}{S}\right)\right)\right],\\
I_4 &:= e^{-rT}\left(\frac{S}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[\Phi\left(\delta_-\left(T,\frac{B^2}{KS}\right)\right) - \Phi\left(\delta_-\left(T,\frac{B}{S}\right)\right)\right],\\
\delta_\pm(T,u) &:= \frac{1}{\sigma\sqrt{T}}\left[\ln u + \left(r \pm \frac{1}{2}\sigma^2\right)T\right].
\end{aligned}$$

- Extensions: Kunitomo and Ikeda (1992), Carr (1995), Boyle and Tian (1998), Davydov and Linetsky (2001), Kou and Wang (2003, 2004).

[10] See also Shreve (2004).

- In practice, many (if not most) contracts with barrier provisions specify discrete monitoring instants, typically daily closings.

- The option is an exotic *partial barrier option* written on an FX rate. The current value of underlying FX rate $S_0 = 1.5$ (i.e. 1.5 units of domestic buys 1 unit of foreign). It matures in one year, i.e. $T = 1$.
- The option knocks out, if the FX rate
  1. is greater than an upper level U in the period between between 1 month's time and 6 month's time; or,
  2. is less than a lower level L in the period between 8^{th} month and 11^{th} month; or,
  3. lies outside the interval $[1.3, 1.8]$ in the final month up to the end of year.



- If it has not been knocked out at the end of year, the owner has the option to buy 1 unit of foreign for X units of domestic, say $X = 1.4$, then, the payoff is $\max\{0, S_T - X\}$.

- We assume that, FX rate follows a geometric Brownian motion

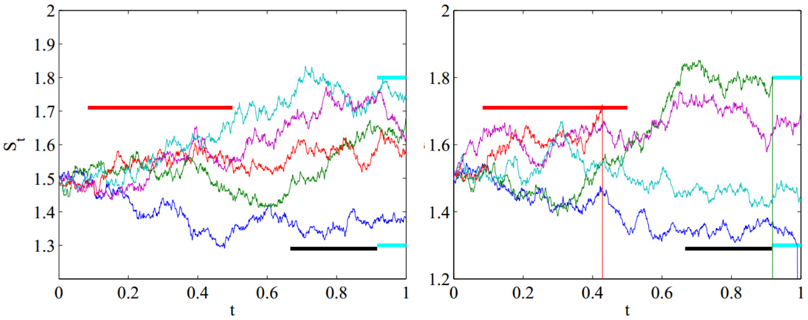
$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \tag{35}$$

where under risk-neutrality $\mu = r_f - r = 0.03$ and $\sigma = 0.12$.

- To simulate path, we divide the time period $[0, T]$ into N small intervals of length $\Delta t = T/N$, and discretize the SDE above by *Euler approximation*

$$S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} Z_t, \quad Z_t \sim \mathcal{N}(0, 1). \tag{36}$$

- The algorithm for pricing this barrier option by Monte Carlo simulation is as described as follows:
  1. Initialize S_0;
  2. Take $S_{i\Delta t}$ as known, calculate $S_{(i+1)\Delta t}$ using equation the discretized SDE as above;
  3. If S_{i+1} hits any barrier, then set payoff to be 0 and stop iteration, otherwise, set payoff at time T to $\max\{0, S_T - X\}$;
  4. Repeat the above steps for M times and get M payoffs;
  5. Calculate the average of M payoffs and discount at rate μ;
  6. Calculate the standard deviation of M payoffs.

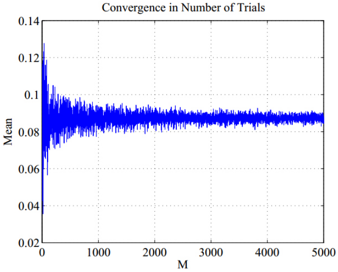


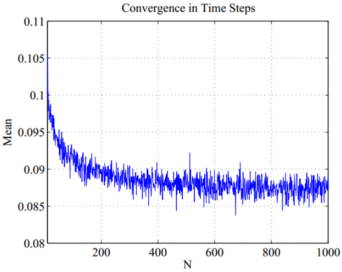
- **Fig.(Left):** all sample paths "survive" (i.e. none of them touches barrier before maturity).
- **Fig.(Right):** some paths touch the barrier before maturity so that payoffs of these paths become zero; if we let τ^* denote the first time the path hits barrier, then $\{S_t\}_{\tau^*<t\leq T} \equiv 0$.
- The accuracy of price estimated by simulations depends on both number of trials M and number of discretization-time steps N. The optimum of efficiency for Euler approximation algorithm is when $N = \alpha\sqrt{M}$ (Duffie and Glynn, 1995).

- We take time steps $N = 1,000$, number of sample paths $M = 10,000$, $S_0 = 1.5$, $n = 1$, $m = 1$, $U = 1.71$, $L = 1.29$, $\mu = 0.03$, $\sigma = 0.12$, $X = 1.4$, the following table lists the result by running code for 10 times:

	1	2	3	4	5	6	7	8	9	10
Mean	0.08531	0.08637	0.08737	0.08811	0.08876	0.08819	0.08656	0.08674	0.08708	0.08769
SE	1.029E-3	1.03E-3	1.04E-3	1.05E-3	1.04E-3	1.04E-3	1.03E-3	1.04E-3	1.04E-3	1.03E-3

- By averaging the mean values above, we get the estimated option price 0.08722.





- The standard errors tend to be 0 as $N \to \infty$, so this method is adoptable.

- **Straddle**: involves one call and one put with the same strike price and expiration date
- **Calendar spread**: combination of options with the same strike price but different expiration dates
- **Butterfly spread**: involves three options of the same type with different strike prices

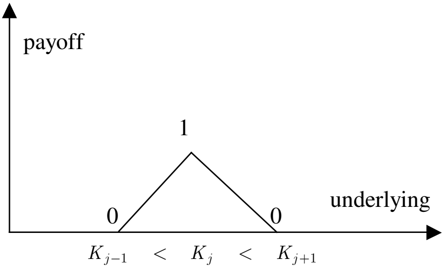


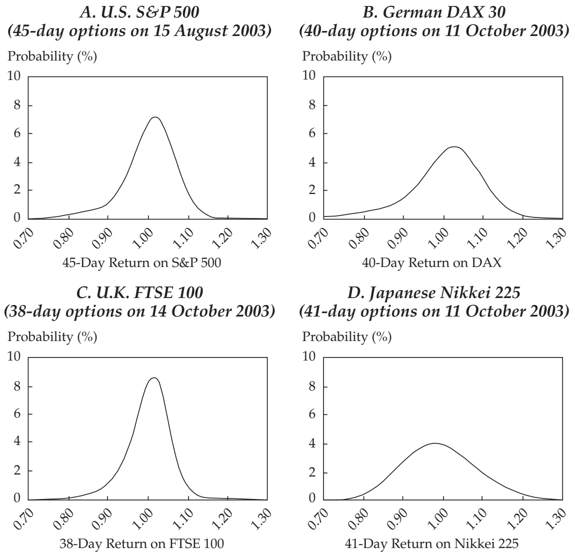
Figure: Approximation for Arrow-Debreu (Arrow and Debreu, 1954) security price $\pi(K_j; T)$

- Forward/future prices only tell about the risk-neutral expectation of the underlying price, as, in general,

$$F_t = \mathbb{E}^{\mathbb{Q}}[S_T \mid S_t],$$

  see Meese and Rogoff (1983), Shiller et al. (1983), Fama (1984b), Fama (1984a), Mankiw et al. (1984), Fama and Bliss (1987), Hardouvelis (1988).

- The "model-free" *implied risk-neutral probability density function* (RND) of the underlying price at the maturity is the second derivative of the market price of European call with respect to the strike price **(Breeden and Litzenberger, 1978)**, see Banz and Miller (1978), Jackwerth and Rubinstein (1996), Aït-Sahalia and Lo (1998), Jackwerth (1999, 2000).



Notes: Returns reported as 1 plus the rate of return.

- Forward-looking analysis for monetary policy of central banks: Malz (1996, 1997), Bahra (1997), Campa et al. (1997), Söderlind and Svensson (1997), Kitsul and Wright (2013), David and Veronesi (2014).
- Event study: Gemmill (1992), Melick and Thomas (1997), Jondeau and Rockinger (2000), Coutant et al. (2001), Galati et al. (2005), Beber and Brandt (2006), Fatum and Hutchison (2006).
- Risk measurement/management: Chang et al. (2012), Buss and Vilkov (2012), Duan and Zhang (2014).
- Portfolio selection/optimisation: DeMiguel et al. (2013), Kempf et al. (2015).

- Implied risk-neutral moments of the underlying: Carr and Madan (2001b), Bakshi et al. (2003).
- Implied risk-neutral underlying price processes/dynamics: Britten-Jones and Neuberger (2000)[11] Jiang and Tian (2005), Carr and Wu (2009), Bollerslev et al. (2009).

[11] Britten-Jones and Neuberger (2000) find that, even initially given a continuum collection of both strikes and maturities of European call/put options, the implied underlying price process is not unique, hence the consistent prices of other associated exotic derivatives are not unique as well!

- Is volatility tradable (Neuberger, 1990)?
- In 1993, CBOE introduced *volatility index* (old) VIX – the average of B-S implied volatilities from 30-calendar-day ATM S&P 100 index (OEX) options[12] – the benchmark index to measure the aggregate volatility of US equity market.
- VIX is also known as investor's "Fear Index" (Whaley, 2000).
- VIX (new) redefined in 2003[13] is a model-free measure of market's expected volatility conveyed by all available OTM S&P 500 index (SPX) options over the next 30-calendar-day period.

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[12] Academics and practitioners often regard ATM B-S implied volatility as an approximate forecast for *realized volatility*, based on empirical evidences by, e.g. Latane and Rendleman (1976), Chiras and Manaster (1978), Canina and Figlewski (1993), Lamoureux and Lastrapes (1993), Christensen and Prabhala (1998).

[13] Ticker for old VIX is switched to VXO, see their key differences in Carr and Wu (2006). New definition of VIX is based on the finding of Carr and Madan (2001b) that a *variance swap* in theory can be replicated by a *static hedging* in a continuum of European options and a *dynamic hedging* in the underlying.

- CBOE's **VIX** (CBOE, 2009) at time t, VIX_t, is defined via **VIX squared**

$$\mathrm{VIX}_t{}^2 := \left[\frac{2}{\tau}\sum_i e^{r\tau}\frac{O_t(K_i,T)}{K_i^2}\Delta K_i - \frac{1}{\tau}\left(\frac{F_t(T)}{K_0}-1\right)^2\right] \times 100^2,$$

where
  - $\tau := 30/365$ (i.e. 30 calendar days);
  - r is time-t risk-free rate with time-to-maturity τ;
  - $T := t + \tau$ is common expiry date for all options;
  - $F_t(T)$ is time-t 30-day *forward SPX* derived from index option prices via *put-call parity*, i.e. $F_t(T) = e^{r\tau}\left[C_t(K,T) - P_t(K,T)\right] + K$;
  - K_0 is the first strike below *forward SPX* level $F_t(T)$;
  - K_i is strike price of i^{th} OTM option $O_t(K_i,T)$ written on SPX: a call $C_t(K_i,T)$ if $K_i > K_0$ and a put $P_t(K_i,T)$ if $K_i < K_0$; both put and call if $K_i = K_0$;
  - $O_t(K_i,T)$ is time-t mid-quote price of each option at strike K_i;
  - ΔK_i is interval between two strikes – half the difference between strike on either side of K_i, i.e. $\Delta K_i := (K_{i+1} - K_{i-1})/2$.

- VIX_t^2 is a linear portfolio of particularly selected options.

- Without specifying the dynamics of underlying SPX index process S_t (i.e. model-free), VIX_t^2 is exactly the discretisation of risk-neutral expectation of a **log contract** (Neuberger, 1994) [14] price at time t (Gatheral, 2006), i.e.

$$\mathrm{VIX}_t^2 \;\to\; -\frac{2}{\tau}\mathbb{E}^{\mathbb{Q}}\left[\ln\left(\frac{S_T}{F}\right) \;\middle|\; \mathcal{F}_t\right] \times 100^2, \qquad \forall\, \Delta K_i \to 0,$$

where

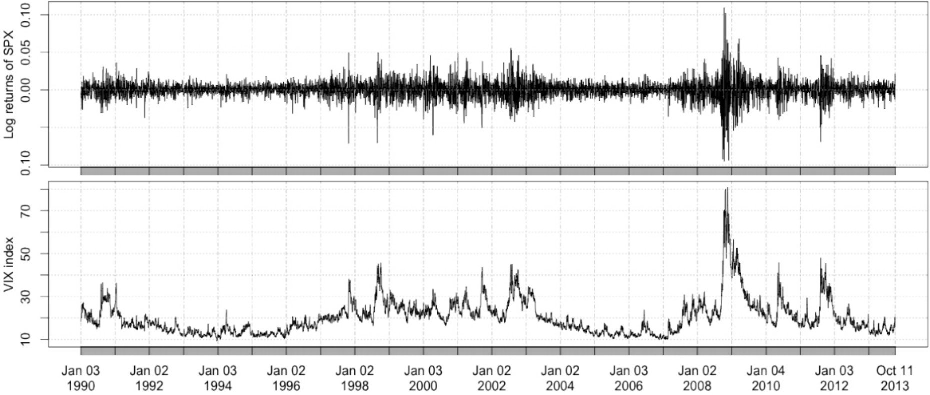
$$F := F_t(T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = e^{r\tau} S_t.$$

[14] *Log contract* was first introduced by Neuberger (1994) for hedging volatility, and laid the theoretical foundation for pricing and trading volatility derivatives later in practice.

§6 VIX

VIX Spot: Volatility Measure

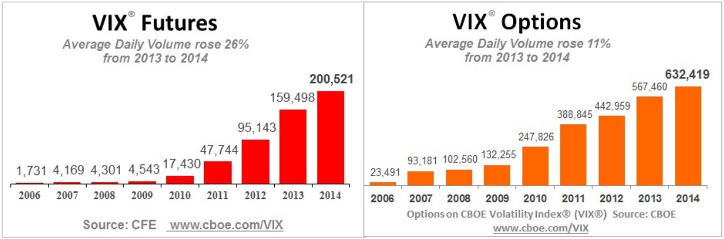
- VIX is a measure of volatility.
- Typical feature of volatility: stochastic, mean-reverting and cluttering (i.e. large moves follow large moves and small moves follow small moves).



Strong Negative Correlation between VIX Spot and Other Indices

Correlation of Monthly Returns (Jan. 31, 1990 - June 30, 2004)									
	CBOE Volatility Index	S&P 500	CBOE S&P 100 Volatility Index	CBOE S&P 500 BuyWrite Index	Russell 2000	Nasdaq 100	Dow 30	EAFE (in US $)	Citigroup 30-yr Treasury
	VIX	SPX	VXO	BXM	RUT	NDX	DJX	EAFE	30-Yr Tr
VIX	1.00								
SPX	-0.62	1.00							
VXO	0.94	-0.63	1.00						
BXM	-0.68	0.87	-0.72	1.00					
RUT	-0.58	0.73	-0.56	0.69	1.00				
NDX	-0.53	0.81	-0.51	0.72	0.77	1.00			
DJX	-0.58	0.93	-0.62	0.83	0.65	0.67	1.00		
EAFE	-0.39	0.64	-0.40	0.54	0.55	0.51	0.64	1.00	
30-Yr Tr	-0.01	0.10	-0.02	0.05	0.01	0.03	0.03	0.04	1.00

- Volatility becomes a tradable asset class: *VIX futures*, Euro-type *VIX options* were launched in 2004, 2006, respectively; they are nowadays among the most actively traded contracts at CBOE, as they are designed deliver pure volatility exposure alone in single and efficient package.



- They provide "catastrophe hedging" tools for stock portfolios, due to negative correlation between VIX and SPX; they could also be used as a hedging vehicle within fixed income, given high correlation between credit spreads and volatility (Carr and Lee, 2009).

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