Lecture: Financial Modelling

– Rate

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Outline

Example (No-arbitrage Pricing Default-free Coupon Bond)

- Suppose 5 default-free *zero-coupon bonds* with nominal \$100 in bond market:

Series	Maturity (Year)	Price (\$)
A_1	1	95
A_2	2	87
A_3	3	80
A_4	4	76
A_5	5	70

- What is the price of a 5-year default-free coupon-bearing bond with nominal \$10,000 and annual coupon rate of 5%?

Year	Cash Flow (\$)
1	500
2	500
3	500
4	500
5	10,500

Example (No-arbitrage Pricing Default-free Coupon Bond)

- Strategy for the issuer of this coupon-bearing bond:
  - Let the current price be $\$B$.
  - Buy 5 of each zero-coupon bonds and another 100 of A_5 series.
  - Cash flows (CF) then are:

Year	Strategy CF ($)	Issue CF ($)	**Net CF** ($)
0	$-5 \times (95 + 87 + 80 + 76 + 70) - 100 \times 70 = -9,040$	B	$B - 9,040$
1	500	-500	0
2	500	-500	0
3	500	-500	0
4	500	-500	0
5	10,500	-10,500	0

- Under no-arbitrage assumption, the current price has to be $B = \$9,040$.

- In general, the pricing formula for *coupon-bearing bonds* in terms of discount factors/rates:

$$B = \sum_{t=1}^{T} \frac{c}{(1+r_t)^t} + \frac{N}{(1+r_T)^T}, \tag{1}$$

  where
  - r_t is interest rate (*zero rate*) of default-free zero-coupon bond with maturity t,
  - c is coupon,
  - N is nominal.

- Or, equivalently,

$$B = \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{N}{(1+y)^T}, \tag{2}$$

  where $y = y_T$ is yield (i.e. *yield to maturity*).

Example (Pricing Default-free Coupon Bond)

Consider a 5-year bond with a yield of 11% (continuously compounded) paying an 8% coupon at the end of each year.

- What is the bond price?

- By using yield $y = 11\%$, we can compute bond price:

$$\begin{aligned} B &= e^{-y\times 1}c + e^{-y\times 2}c + e^{-y\times 3}c + e^{-y\times 4}c + e^{-y\times 5}(c+N) \\ &= e^{-0.11\times 1}8 + e^{-0.11\times 2}8 + e^{-0.11\times 3}8 + e^{-0.11\times 4}8 + e^{-0.11\times 5}(8+100) \\ &= 86.8. \end{aligned}$$

- Yield y may changes throughout the time.

- Changing interest rates creates one of major risk sources for banks, insurance companies, and other financial institutions, and any interest-sensitive security trading.
- The origins of risk are from changes in interest rates (discount rates, or equivalently, yields).
- Mainly due to the high (infinite) dimensionality, it is more difficult to manage than the risk arising from market variables (e.g. equity, commodity prices).

- There are many different interest rates in any given currency (e.g. treasury rates, inter-bank borrowing and lending rates, swap rates, mortgage rates, deposit rates, prime borrowing rates, etc.), which tend to move together but not perfectly correlated.
- The gross notional amount outstanding in OTC interest-rate and FX contracts totalled $563.293 trillion and $74.782 trillion respectively in Dec 2014.
- We need a term structure (more than a single number) or yield curve to fully describe the interest rate environment, i.e. a function describing the variation of rate with maturity (time to maturity).

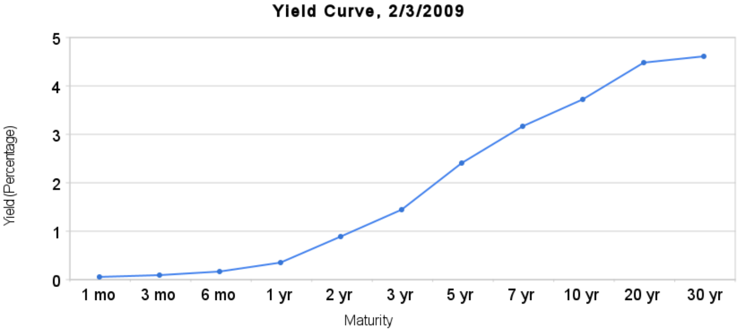


Figure: Linearly Interpolated Yield Curve

- Standard interpolation for yield curve is cubic-spline algorithm[1] (Longstaff et al., 2005).

[1] Spline algorithm: a piecewise-polynomial real function that passes through a given set of control points.

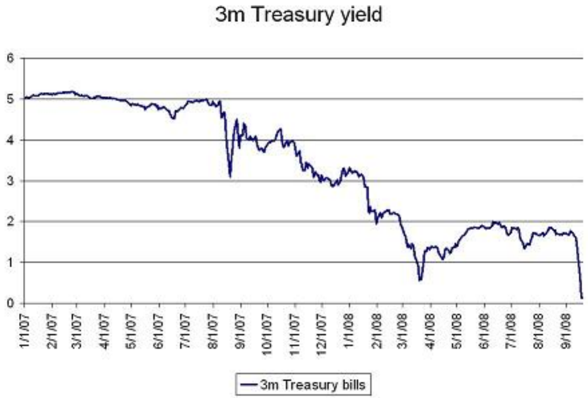


Figure: Time Series of 3-month Yield of Treasury Bill

- Most widely used model for yield movement may be Nelson-Siegel approach (Nelson and Siegel, 1987).

- Traditionally, traders has assumed that LIBOR/swap zero curve is risk-free zero curve.
- Treasury curve is about 50 basis points below LIBOR/swap zero curve.
- Treasury rates are considered to be artificially low for a variety of regulatory and tax reasons:
  - Treasury bills/bonds must be purchased by financial institutions to fulfil regulatory requirements;
  - The capital a bank is required to hold for investment in Treasury bills/bonds is substantially smaller than the one in other very-low-risk instruments;
  - Treasury instruments are given a favourable tax treatment.

- LIBOR (London interbank offered rate): average interest rate estimated by leading banks in London that they would be charged if borrowing from other banks. LIBOR rates are 1-, 3-, 6-, and 12-month borrowing rates for companies that have a AA rating.
  - provided by British Bankers' Association (BBA) in different currencies for maturities ranging from overnight to 12-months at 11 a.m. every business day;
  - lowest and highest quartiles reported are discarded;
  - many loans to corporations and governments, as well as some mortgages, have floating rates that are reset to LIBOR periodically;
  - at least $350 trillion (vs US GDP $15 trillion) in derivatives and other financial products are linked to the Libor.

- “This dwarfs by orders of magnitude any financial scam in the history of markets” – Andrew Lo, MIT Professor of Finance.
- A series of manipulations connected to the LIBOR.
- Banks were falsely inflating or deflating their rates so as to profit from trades, or to give the impression that they were more.
- BBA said on 25 September 2012 that it would transfer oversight of LIBOR to UK regulators.

- A key risk management activity for a bank is managing *net-interest income* (i.e. the excess of interest received over interest paid).
- Most banks have asset-liability management (ALM) groups to ensure that *net interest margin* (*net-interest income* divided by *income-producing assets*) remains roughly constant through time.

- Suppose that, market's best guess is that future short term rates will equal today's rates (i.e. martingale). What would happen if a bank posted the following rates?

Maturity (yrs)	Deposit Rate	Mortgage Rate
1	**3%**	6%
5	3%	**6%**

  - Most consumers choose 1Y deposit (rather than 5Y) as more financial flexibility;
  - most consumers choose 5Y mortgage (rather than 1Y) as less refinancing risk;
  - this needs rolling over deposits, and causes the risk of asset/liability mismatch of maturities for bank;
  - this opens to interest rate risk: if interest rate (deposit rate) increases, net-interest income declines.

- How can bank manage its risks?

Maturity (yrs)	Deposit Rate	Mortgage Rate
1	3%	6%
5	**4%**↑	**7%**↑

- A bank that funds long-term loans with short-term deposits has to replace maturing deposits with new deposits on a regular basis (i.e. *rolling over* deposits).
- When long-term loans are funded with short-term deposits, interest rate swaps can be used to hedge interest-rate risk, but this does not hedge liquidity risk.
- **Rollover risk** (He and Xiong, 2012): if depositors lose confidence in bank, bank might find it difficult to do rolling over deposits (e.g. Northern Rock, Bear Stearns, Lehman Brothers).
- This caused the major problem of US' repo market during 2008's crisis (Gorton and Metrick, 2012).

- Direct assessment – full valuation: recalculate whole portfolio value for every possible changes in yield curves.
  - It is usually very computationally intensive;
  - Consider only the most likely scenarios;
  - Banks are usually required to simulate possible changes in their portfolio values, typical changes to be considered are 200 bp up and down uniformly;
  - Consider historical scenarios;
  - Determine the risk factors that account for the most of variation in changes (usually via *principal component analysis – PCA*).

- Scenario analysis:
  1. Parallel shifts in yield curves;
  2. Non-parallel shifts in yield curves.

- Duration is a widely used measure of a bond or portfolio's exposure to yield curve movement.
- Duration D of bond

$$D := -\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}y}, \qquad \frac{\Delta B}{B} \approx -D\Delta y, \tag{3}$$

  where
  - B is bond's market price;
  - y is its yield (continuously compounded);
  - Δy is a small change in bond's yield,

$$\Delta B := B(y+\Delta y) - B(y). \tag{4}$$

- Duration measures the sensitivity of percentage changes in bond's price to changes in its yield.

- Market price of a bond in general that provides cash flow c_i at time t_i is

$$B = \sum_{i=1}^{n} c_i e^{-y t_i}. \tag{5}$$

- Duration of this bond is

$$D = \sum_{i=1}^{n} \left(\frac{c_i e^{-y t_i}}{B} \right) t_i, \tag{6}$$

where weight $w_i := \frac{c_i e^{-y t_i}}{B}$ is the ratio of present value of cash flow c_i at time t_i to bond price B.

- Duration is a measure of how long the bondholder has to wait for cash flows.
  - e.g. a zero-coupon bond of n years has a duration of n years;
  - e.g. a coupon-bearing bond of n years has a duration of less than n years.

Example (Bond Duration)

- Calculate the duration for a 3-year bond paying a coupon 10% (semi-annually), with face value=$100, bond yield=12%:

Time t_i (yrs)	Cash Flow c_i ($)	PV_i ($)	Weight w_i	Time $\times$ Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total		B=94.213	1.000	D=2.653

- If yield y is expressed with *periodically compounding m* times per year (rather than *continuously compounding*), then, we have bond price in general

$$B = \sum_{i=1}^{n} \frac{c_i}{(1 + y/m)^{mt_i}}, \tag{7}$$

and

$$\frac{\Delta B}{B} \approx -D^* \Delta y, \qquad D^* := \frac{D}{1 + y/m}, \tag{8}$$

where D^* is referred to as the *modified duration*, D is called *Macaulay duration*.

- $D^* = D$ if yield is expressed continuously compounded.

- *Dollar duration* of a bond is defined as the product of its duration and its price

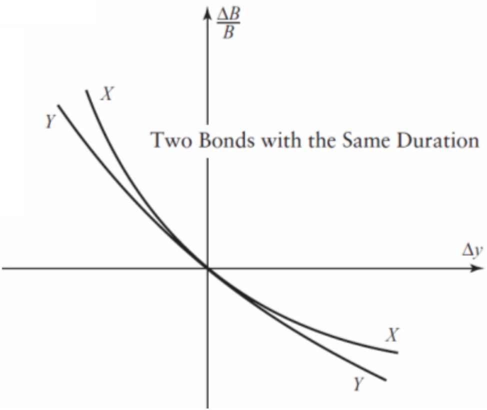
$$D_{\$} := -\frac{\mathrm{d}B}{\mathrm{d}y}, \tag{9}$$

$$\Delta B \approx -D_{\$}\Delta y. \tag{10}$$

- Dollar duration relates actual changes in bond's price to its yield.
- Dollar duration is similar to delta measure.
- *DV01*: dollar duration multiplied by 0.0001, measures the impact of a one-basis-point increase in all rates.

- Duration measures bond's exposure to a small change Δy in yield

$$D \approx -\frac{\frac{\Delta B}{B}}{\Delta y}.$$



- Convexity of a bond (measuring curvature) is defined as

$$C := \frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}. \tag{11}$$

- By Taylor expansion, second-order approximation to the change in bond price is

$$\Delta B := B(y+\Delta y) - B(y) \approx \frac{\mathrm{d}B}{\mathrm{d}y}\Delta y + \frac{1}{2}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2}(\Delta y)^2,$$

i.e.

$$\frac{\Delta B}{B} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2.$$

- Duration and convexity can be defined similarly for portfolios of bonds and other interest-rate dependent securities,

$$\frac{\Delta P}{P} \approx -D_P \Delta y, \tag{12}$$

where Δy is the size of small parallel shift.

- Duration of a portfolio is the weighted (=individual price/total price) average of durations of components of portfolio, similarly for convexity,

$$D_P = \sum_{i=1}^{n} w_i D_i, \qquad C_P = \sum_{i=1}^{n} w_i C_i. \tag{13}$$

- We define a parallel shift in yield curve as a shift where all interest rates change by the same amount.

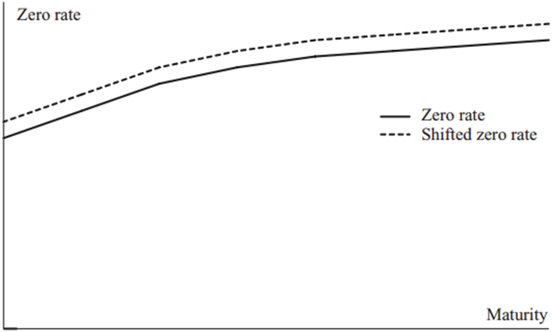
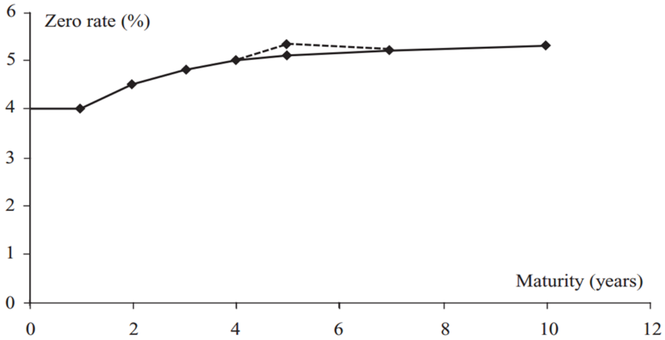


FIGURE 8.2 A Parallel Shift in Zero Rates

- Duration measures the effect of a small parallel shift in yield curve.
- Duration plus convexity measure the effect of a larger parallel shift in yield curve.
- However, they do not measure the effect of non-parallel shifts.
- *Portfolio immunization*: a portfolio (consisting of long and short positions in interest-rate-dependent assets) can be protected against relatively small or large parallel shift in yield curve.

- A *partial duration* calculates the effect on a portfolio of a change to just one point on zero curve.
    - e.g. the 5^{th} point on zero curve is shifted, the other points are not shifted, and rates on the shifted yield curve are calculated using linear interpolation.



- The *partial duration* of portfolio for the i^{th} point on zero curve is

$$D_i :\approx -\frac{1}{P}\frac{\Delta P_i}{\Delta y_i}, \tag{14}$$

where

  - Δy_i is the size of small change made to the i^{th} point on yield curve;
  - ΔP_i is the resulting change in portfolio value, i.e.

$$\Delta P_i := P(y_1, ..., y_i + \Delta y_i, ..., y_n) - P(y_1, ..., y_i, ..., y_n).$$

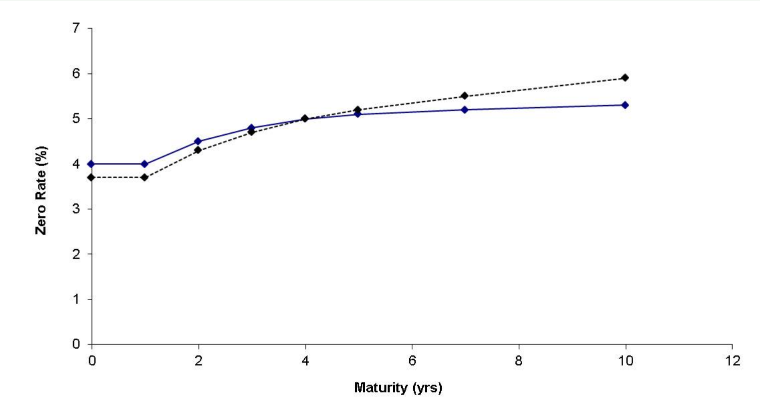
- The sum of all partial durations equals to usual duration:

$$D = \sum_i D_i.$$

- Partial durations can be used to investigate the impact of any yield curve change, as any yield curve change can be defined in terms of changes to individual points on yield curve.

Example (Rotation)

- Define a rotation (or twist), we could change the 1-, 2-, 3-, 4-, 5-, 7, and 10-year maturities by $-3e$, $-2e$, $-e$, 0, e, $3e$, $6e$ for some small e (e.g. 1bp):



Example (Rotation)

Table: Partial Durations for a Portfolio

Maturity (years), i	1	2	3	4	5	7	10	Total
Duration, D_i	0.2	0.6	0.9	1.6	2.0	-2.1	-3.0	0.2

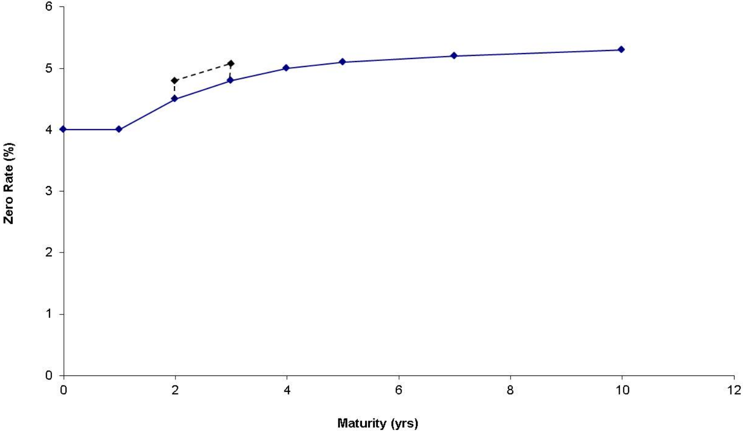
- Impact of this rotation on the proportional change in the portfolio value is

$$\begin{aligned}\frac{\Delta P}{P} &\approx \sum_{i=1}^{n}\big(-D_i\Delta y_i\big)\\ &= -\Big[0.2\times(-3e)+0.6\times(-2e)+0.9\times(-e)\\ &\quad +1.6\times 0+2.0\times e-2.1\times 3e-3.0\times 6e\Big]\\ &= 25.0e.\end{aligned}$$

- For a parallel shift of e in yield curve, i.e. $\Delta y_i \equiv e$,

$$\frac{\Delta P}{P} \approx -D\Delta y = -0.2e.$$

- *Bucketing approach*: bucket the yield curve and investigate the effect of a small change to each bucket (segment):



- The evolution of interest rate curve can be split into three standard movements:

  1. **Shift movement** (changes in level) accounts for 70% to 80% of observed movements on average;
  2. **Twist movement** (changes in slope) accounts for 15% to 30% of observed movements on average;
  3. **Butterfly movement** (changes in curvature) accounts for 1% to 5% of observed movements on average.

Example (Duration: Small Yield Change)

Consider a 5-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.

1. What is bond price?
2. What is bond duration?
3. Use duration to calculate the effect on bond price of a 0.2% decrease in its yield.
4. Recalculate the exact bond price on the basis of a 10.8% per annum yield, and verify that the result is in agreement with your answer to (3).

- By using yield y, we can compute the price:

$$B = e^{-0.11\times 1}8 + e^{-0.11\times 2}8 + e^{-0.11\times 3}8 + e^{-0.11\times 4}8 + e^{-0.11\times 5}(100+8) = 86.8$$

- Duration is computed as follows:

$$D = -\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}y} = \frac{1}{B}\sum_{i=1}^{n} t_i c_i e^{-y t_i} = \frac{1}{86.8}\left(\begin{array}{c} e^{-0.11\times 1}8 + 2e^{-0.11\times 2}8 + 3e^{-0.11\times 3}8 \\ +4e^{-0.11\times 4}8 + 5e^{-0.11\times 5}(100+8) \end{array}\right) = 4.256$$

- Using duration, we can calculate the effect on bond's price of a 0.2% decrease in its yield:

$$D = -\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}y} \quad \Rightarrow \quad \Delta B \approx -DB\Delta y = -4.25 \times 86.8 \times (-0.002) = 0.738$$

- Actual change is $B' - B$:

$$\begin{aligned} B' &= e^{-0.108\times 1}8 + e^{-0.108\times 2}8 + e^{-0.108\times 3}8 \\ &\quad + e^{-0.108\times 4}8 + e^{-0.108\times 5}(100+8) \\ &= 87.543, \\ B' - B &= 87.543 - 86.8 = 0.743 \simeq 0.738 \end{aligned}$$

Example (Duration & Convexity: Large Yield Change)

A 6-year bond with a continuously compounded yield of 4% provides a 5% coupon at the end of each year.

1. Use duration and convexity to estimate the effect of a 1% increase in the yield on the bond price.
2. How accurate is the estimate?

- Bond price:

$$B = e^{-0.04\times 1}5 + e^{-0.04\times 2}5 + e^{-0.04\times 3}5 + e^{-0.04\times 4}5 + e^{-0.04\times 5}5 + e^{-0.04\times 6}105 = 104.8$$

- Duration:

$$\begin{aligned} D &= \frac{1}{B}\sum_{i=1}^{n} t_i c_i e^{-yt_i} \\ &= \frac{1}{104.8}\left(\begin{array}{c} e^{-0.04\times 1}5 + 2e^{-0.04\times 2}5 + 3e^{-0.04\times 3}5 \\ +4e^{-0.04\times 4}5 + 5e^{-0.04\times 5}5 + 6e^{-0.04\times 6}105 \end{array}\right) = 5.3476 \end{aligned}$$

- Convexity:

$$\begin{aligned} C &= \frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2} = \frac{1}{B}\sum_{i=1}^{n} t_i^2 c_i e^{-yt_i} \\ &= \frac{1}{104.8}\left(\begin{array}{c} e^{-0.04\times 1}5 + 4e^{-0.04\times 2}5 + 9e^{-0.04\times 3}5 \\ +16e^{-0.04\times 4}5 + 25e^{-0.04\times 5}5 + 36e^{-0.04\times 6}105 \end{array}\right) = 30.603 \end{aligned}$$

- Estimated change:

$$\frac{\Delta B}{B} \approx -D \times \Delta y + \frac{1}{2} C \times \Delta y^2,$$

$$\Delta B \approx \left(-D \times \Delta y + \frac{1}{2} C \times \Delta y^2\right) B$$

$$= \left(-5.3476 \times 0.01 + \frac{1}{2} \times 30.603 \times 0.01^2\right) 104.8 = -5.4439$$

- Exact change:

$$\begin{aligned} B(y+\Delta y) &= e^{-0.05\times 1}5 + e^{-0.05\times 2}5 + e^{-0.05\times 3}5 \\ &\quad + e^{-0.05\times 4}5 + e^{-0.05\times 5}5 + e^{-0.05\times 6}(100+5) \\ &= 99.357 \\ \Delta B = B(y+\Delta y) - B(y) &= 99.357 - 104.8 = -5.443 \end{aligned}$$

- If only duration is used for calculating change, then,

$$\Delta B \approx (-D \times \Delta y) \times B = (-5.3476 \times 0.01)104.8 = -5.6043$$

which makes a larger error.

- r_t is the continuous-time *instantaneous risk-free interest rate* (or *short rate*)[2] at time t (i.e. within period $[t, t + \mathrm{d}t]$), at which *bank account*[3] accrues, i.e.

$$\frac{\mathrm{d}B_t}{B_t} = r_t \mathrm{d}t, \qquad t \geq 0, \tag{15}$$

  or,

$$B_t = B_0 e^{\int_0^t r_u \mathrm{d}u},$$

  where B_t is the value of continuously compounded bank account at time t.

- r_t (theoretically unobservable) is the rate earned on shortest-term loans starting at time t.

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[2] In practice, one takes the yield in a 1-month US Treasury bill, or a comparable short-maturity bond such as overnight rate (e.g. *overnight indexed swaps* -OIS), as a proxy for short rate.

[3] This account is called *rolling money market account*, which is closely related to a zero-coupon bond but with no expiry time.

- The value at time t of default-free *zero-coupon bond* (or *discount bond*) paying $\$1$ at maturity T is

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \,\middle|\, \mathcal{F}_t \right], \qquad t \in [0,T], \tag{16}$$

  where $\mathbb{Q}$ is risk-neutral probability measure, $\mathcal{F}_t$ is information about r_t up to time t.

- The dependence on times t and T reflects term structure of interest rates.

- The continuously compounded *yield* (or *yield-to-maturity*) $y(t,T)$ within period $[t,T]$ can be easily derived from

$$P(t,T) = e^{-y(t,T)(T-t)} \tag{17}$$

  as

$$y(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{18}$$

- If short rate r_t under $\mathbb{Q}$ follows a general *Markovian diffusion process* as

$$\mathrm{d}r_t = \mu(r_t, t)\mathrm{d}t + \sigma(r_t, t)\mathrm{d}W_t, \qquad t \in [0, T],$$

where $\mu(r, t), \sigma(r, t)$ are deterministic functions, $r_0 > 0$, W_t is Brownian motion, then, zero-coupon bond price (process)

$$P(t, T) = V(r, t) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \,\middle|\, r_t = r \right]$$

follows **Vasicek's PDE** (Vasicek, 1977):

$$\frac{\partial}{\partial t}V(r, t) + \mu(r, t)\frac{\partial}{\partial r}V(r, t) + \frac{1}{2}\sigma^2(r, t)\frac{\partial^2}{\partial r^2}V(r, t) - rV(r, t) = 0, \quad t \in [0, T],$$

with terminal condition $V(r, T) = 1$ for all r at time $t = T$.

- We set up a portfolio containing two bonds with different maturities T_1 and T_2. The bond with maturity T_1 has price $V_1(r, t; T_1)$ and the bond with maturity T_2 has price $V_2(r, t; T_2)$. We hold one of the former and a number $-\Delta$ of the latter. We have

$$\Pi = V_1 - \Delta V_2.$$

- The change in this portfolio in a time dt is given by

$$d\Pi = \frac{\partial V_1}{\partial t} dt + \frac{\partial V_1}{\partial r} dr + \frac{1}{2}\sigma^2 \frac{\partial^2 V_1}{\partial r^2} dt - \Delta \left(\frac{\partial V_2}{\partial t} dt + \frac{\partial V_2}{\partial r} dr + \frac{1}{2}\sigma^2 \frac{\partial^2 V_2}{\partial r^2} dt \right).$$

- Choose

$$\Delta = \frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}$$

to eliminates all randomness in dt.

- We then have

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right) \frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_2}{\partial r^2}\right) dt \\ &= r\Pi dt = r\left(V_1 - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}\right) V_2\right) dt, \end{aligned}$$

- Collecting all V_1 terms on the left-hand side and all V_2 terms on the right-hand side we find that

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}}.$$

- The only way for this to be possible is for both sides to be independent of the maturity date. Dropping the subscript from V, we have

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r, t).$$

- It is convenient to write

$$a(r,t) = -\mu(r,t)$$

- The bond pricing equation is therefore

$$\frac{\partial}{\partial t}V(r,t) + \mu(r,t)\frac{\partial}{\partial r}V(r,t) + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2}{\partial r^2}V(r,t) - rV(r,t) = 0, \quad t \in [0,T],$$

- In a time-step dt one bond changes in value by

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + \mu \frac{\partial V}{\partial r} \right) dt + \sigma \frac{\partial V}{\partial r} dW_t$$

- From Vasicek's PDE this may be written as

$$\frac{dV}{V} = r dt + \frac{\sigma}{V} \frac{\partial V}{\partial r} dW_t$$

- **Merton's Model** (Merton, 1973):

$$\mathrm{d}r_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \tag{19}$$

where $\mu, \sigma, r_0 > 0$ are constants, $\{r_t\}_{t \geq 0}$ is a *Brownian motion with drift*.

- Properties:

1. Gaussian process

$$r_t = r_0 + \mu t + \sigma W_t \quad \sim \mathcal{N}\left(r_0 + \mu t, \sigma^2 t\right);$$

2. $\Pr\{r_t < 0\} > 0$;

3. Bond price at time $t = 0$:

$$P(0, T) = \exp\left(-r_0 T - \frac{1}{2}\mu T^2 + \frac{1}{6}\sigma^2 T^3\right).$$

- **Vasicek's Model (Vasicek, 1977)**[4]:

$$\mathrm{d}r_t = \theta(\mu - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{20}$$

where $\theta, \mu, \sigma, r_0 > 0$ are constants. This stochastic process $\{r_t\}_{t \geq 0}$ is called *Gaussian Ornstein-Uhlenbeck (OU) process*[5].

- Properties:
  1. mean-reverting Gaussian process;
  2. $\Pr\{r_t < 0\} > 0$;
  3. $$r_t = \mu + e^{-\theta t}\left[(r_0 - \mu) + \sigma \int_0^t e^{\theta u}\mathrm{d}W_u\right] \quad \sim \mathcal{N};$$
  4. $P(t, T) = e^{A(t,T) - C(t,T)r_t}$ where
  $$A(t, T) = \left(\mu - \frac{\sigma^2}{2\theta^2}\right)\left[C(t, T) - (T - t)\right] - \frac{\sigma^2}{4\theta}C^2(t, T), \quad C(t, T) = \frac{1 - e^{-\theta(T-t)}}{\theta}.$$

[4] Oldrich Vasicek is also co-founder of a San Francisco-based quantitative risk management firm KMV (Stephen Kealhofer, John McQuown, Oldrich Vasicek), now Moody's Analytics.

[5] The continuous-time version of *first-order autoregressive process*.

- **Cox-Ingersoll-Ross (CIR) Model** (Cox et al., 1985)[6]:

$$\mathrm{d}r_t = \theta(\mu - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t, \tag{21}$$

where $\theta, \mu, \sigma, r_0 > 0$ are constants. $\{r_t\}_{t\geq 0}$ is a non-negative stochastic process called *Feller process* (Feller, 1951), or *CIR process* or *square-root process*.

- Properties:
  1. mean-reverting non-Gaussian process;
  2. $P(t, T) = e^{A(t,T)-C(t,T)r_t}$ where

$$A(t,T) = \frac{2\theta\mu}{\sigma^2}\ln\left(\frac{2\gamma e^{(\gamma+\theta)(T-t)/2}}{(\gamma+\theta)(e^{\gamma(T-t)}-1)+2\gamma}\right), \quad C(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\theta)(e^{\gamma(T-t)}-1)+2\gamma},$$

where $\gamma = \sqrt{\theta^2 + 2\sigma^2}$.

[6] Also see Brown and Dybvig (1986), Gibbons and Ramaswamy (1993), Brown and Schaefer (1994); the empirical study by Pearson and Sun (1994) rejected CIR as a good candidate for describing the Treasury market; estimation method in Overbeck and Rydén (1997); two-factor CIR model (Chen and Scott, 1993); generalised CIR process with jumps of CBI model (Filipović, 2001).

1. *Interest Rate Models – Theory and Practice: with Smile, Inflation and Credit* (Brigo and Mercurio, 2007)
2. *Interest Rate Models: An Introduction* (Cairns, 2004)
3. *Modelling Fixed-Income Securities and Interest Rate Options* (Jarrow, 2002)
4. *Interest Rate Modelling* (James and Webber, 2000)
5. *Modern Pricing of Interest-rate Derivatives: The LIBOR Market Model and Beyond* (Rebonato, 2002)
6. *The SABR/LIBOR Market Model: Pricing, Calibration and Hedging for Complex Interest-rate Derivatives* (Rebonato et al., 2011)
7. *Interest Rate Modeling, Volume 1: Foundations and Vanilla Models; Volume 2: Term Structure Models; Volume 3: Products and Risk Management* (Andersen and Piterbarg, 2010)

Brainteaser

- The volatility for a bond is often quoted as yield volatility instead of price volatility, so, provide a formula to convert yield volatility into (bond) price volatility.

References I

Andersen, L. and Piterbarg, V. (2010). *Interest Rate Modeling, Volume 1: Foundations and Vanilla Models; Volume 2: Term Structure Models; Volume 3: Products and Risk Management*. Atlantic Financial Press.

Brigo, D. and Mercurio, F. (2007). *Interest Rate Models – Theory and Practice: with Smile, Inflation and Credit*. Springer.

Brown, R. H. and Schaefer, S. M. (1994). The term structure of real interest rates and the Cox, Ingersoll, and Ross model. *Journal of Financial Economics*, 35(1):3–42.

Brown, S. J. and Dybvig, P. H. (1986). The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates. *The Journal of Finance*, 41(3):617–630.

Cairns, A. J. (2004). *Interest Rate Models: An Introduction*. Princeton University Press.

Chen, R.-R. and Scott, L. (1993). Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates. *The Journal of Fixed Income*, 3(3):14–31.

Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Feller, W. (1951). Two singular diffusion problems. *The Annals of Mathematics*, 54(1):173–182.

Filipović, D. (2001). A general characterization of one factor affine term structure models. *Finance and Stochastics*, 5(3):389–412.

Gibbons, M. R. and Ramaswamy, K. (1993). A test of the Cox, Ingersoll, and Ross model of the term structure. *Review of Financial Studies*, 6(3):619–658.

References II

Gorton, G. and Metrick, A. (2012). Securitized banking and the run on repo. *Journal of Financial Economics*, 104(3):425–451.

He, Z. and Xiong, W. (2012). Rollover risk and credit risk. *The Journal of Finance*, 67(2):391–430.

James, J. and Webber, N. (2000). *Interest Rate Modelling*. Wiley-Blackwell.

Jarrow, R. A. (2002). *Modelling Fixed-Income Securities and Interest Rate Options*. Stanford University Press.

Longstaff, F. A., Mithal, S., and Neis, E. (2005). Corporate yield spreads: Default risk or liquidity? new evidence from the Credit Default Swap market. *The Journal of Finance*, 60(5):2213–2253.

Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183.

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *The Journal of Business*, 60(4):473–489.

Overbeck, L. and Rydén, T. (1997). Estimation in the Cox-Ingersoll-Ross model. *Econometric Theory*, 13(3):430–461.

Pearson, N. D. and Sun, T. (1994). Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model. *The Journal of Finance*, 49(4):1279–1304.

Rebonato, R. (2002). *Modern Pricing of Interest-rate Derivatives: The LIBOR Market Model and Beyond*. Princeton University Press.

Rebonato, R., McKay, K., and White, R. (2011). *The SABR/LIBOR Market Model: Pricing, Calibration and Hedging for Complex Interest-rate Derivatives*. John Wiley & Sons.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.