Lecture: Financial Modelling

– Credit

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Outline

§1 Default Risk

Credit Risk

- Credit risk arises from the possibility that borrowers, bond issuers, and counterparties in transactions may default.
- It exists in commercial banking (e.g. credit cards, loans), investment banking (e.g. corporate bonds, credit derivatives), sovereign (e.g. Argentina, Russian, Greece).

- Regulators require banks to keep capital for credit risk.
- Under Basel II, banks can, with approval from bank regulators, develop their own models to estimate default probabilities for determining the amount of capital they are required to keep.
- This leads banks to search for various approaches of estimating default probabilities:
    - Use accounting data (e.g. Altman's Z-score);
    - Use historical default data (e.g. from Moody's);
    - Use bond prices from the market;
    - Use Credit Default Swap (CDS) spreads from the market;
    - Use equity prices from the market (e.g. Merton's structure model).

§2 Accounting-Based Credit Risk Modelling

Altman's Z-score for the Publicly-listed Firms

- Altman's Z-score model (Altman, 1968) based on *discriminant analysis* predicts defaults of publicly-traded manufacturing companies (e.g. Toyota, Volkswagen, Samsung Electronics) within 2 years from 5 firm-specific accounting ratios:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5, \tag{1}$$

  where

  - $X_1 =$ Working Capital / Total Assets (measuring liquid assets relative to the size of company), Working Capital = Current Assets – Current Liabilities;
  - $X_2 =$ Retained Earnings / Total Assets (measuring cumulative profitability over time that reflects earning power and firm's age);
  - $X_3 =$ Earnings Before Interest and Taxes (EBIT) / Total Assets (measuring productivity and operating efficiency of the firm's assets, abstracting from any tax or leveraging factors);
  - $X_4 =$ Market Value of Equity / Book Value of Liabilities (measuring how far firm's assets can decline before the company becomes insolvent);
  - $X_5 =$ Sales / Total Assets (measuring ability of firm's assets to generate sales).

- Prediction – the greater a firm's bankruptcy potential within 2 years, the lower its overall index Z-score (discriminant score):
    - If $Z > 3.0$, safe – default is unlikely;
    - If $2.7 < Z < 3.0$, we should be on alert;
    - If $1.8 < Z < 2.7$, there is a moderate chance of default;
    - If $Z < 1.8$, financial distress – there is a high chance of default.

Altman's Z-score for the Privately-held: Discriminant Analysis

- Z-score model predicts defaults of privately-held firms (e.g. Cargill, PWC, Ernst & Young) from 5 accounting ratios:

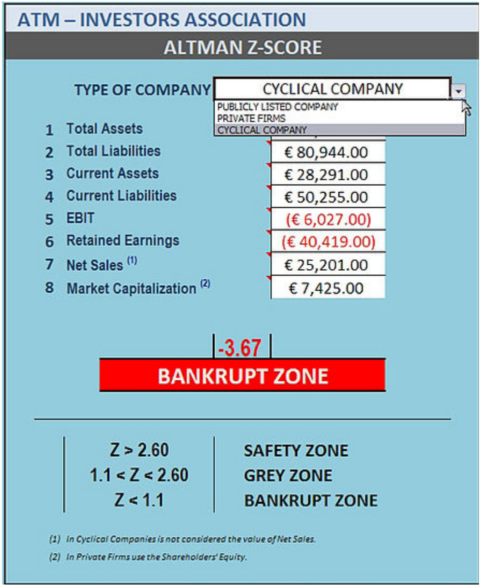
$$Z = 0.717X_1 + 0.847X_2 + 3.107X_3 + 0.420X_4 + 0.998X_5, \qquad (2)$$

  where
  - $X_1 =$ (Current Assets – Current Liabilities) / Total Assets;
  - $X_2 =$ Retained Earnings / Total Assets;
  - $X_3 =$ Earnings Before Interest and Taxes (EBIT) / Total Assets;
  - $X_4 =$ Book Value of Equity / Total Liabilities;
  - $X_5 =$ Sales / Total Assets.

- Prediction:
  - If $Z > 2.9$, safe – default is unlikely;
  - If $Z < 1.23$, financial distress – there is a high chance of default.

§2 Accounting-Based Credit Risk Modelling

Altman's Z-score Calculator



ATM – INVESTORS ASSOCIATION

ALTMAN Z-SCORE

TYPE OF COMPANY: CYCLICAL COMPANY

1	Total Assets	
2	Total Liabilities	€ 80,944.00
3	Current Assets	€ 28,291.00
4	Current Liabilities	€ 50,255.00
5	EBIT	(€ 6,027.00)
6	Retained Earnings	(€ 40,419.00)
7	Net Sales (1)	€ 25,201.00
8	Market Capitalization (2)	€ 7,425.00

-3.67

BANKRUPT ZONE

$Z > 2.60$	SAFETY ZONE
$1.1 < Z < 2.60$	GREY ZONE
$Z < 1.1$	BANKRUPT ZONE

(1) In Cyclical Companies is not considered the value of Net Sales.

(2) In Private Firms use the Shareholders' Equity.

§3 Intensity-Based Credit Risk Modelling

Credit Ratings

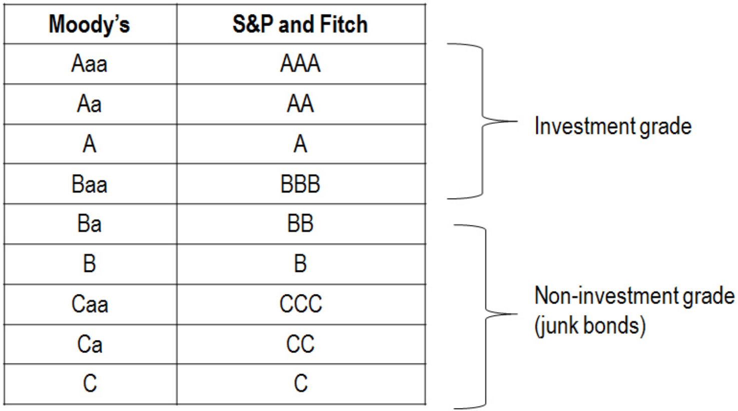
- Credit ratings measure the creditworthiness of (corporate or sovereign) debt instruments (e.g. bonds, CDSs, CDO tranches).
    - They are widely used by financial institutions and regulators for trading, pricing and risk management.
    - They change relatively infrequently for rating stability;
    - They change only when there is reason to believe that a long-term change in the company's creditworthiness has taken place.
- Three major global rating agencies: Moody's, Standard&Poor, Fitch Rating.

STANDARD
&POOR'S

FitchRatings
KNOW YOUR RISK

§3 Intensity-Based Credit Risk Modelling

Credit Ratings



Moody's	S&P and Fitch
Aaa	AAA
Aa	AA
A	A
Baa	BBB
Ba	BB
B	B
Caa	CCC
Ca	CC
C	C

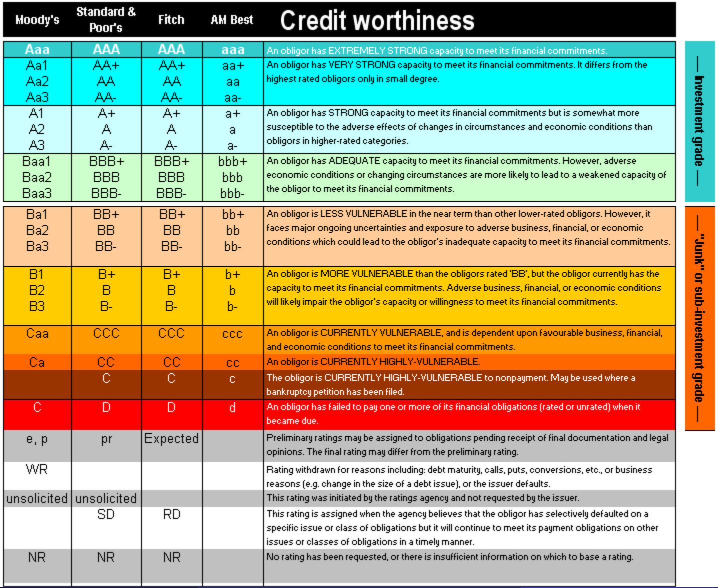
Investment grade

Non-investment grade (junk bonds)

Figure: Notation Systems of Credit Ratings

§3 Intensity-Based Credit Risk Modelling

Subdivisions of Credit Ratings for Finer Rating Measure



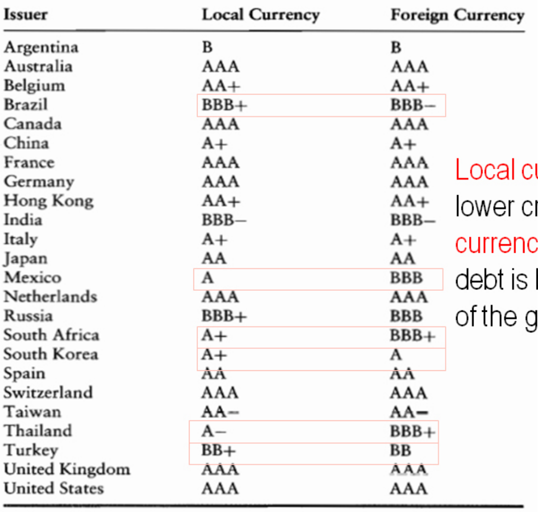
Moody's	Standard & Poor's	Fitch	AM Best	Credit worthiness
Aaa	AAA	AAA	aaa	An obligor has EXTREMELY STRONG capacity to meet its financial commitments.
Aa1 Aa2 Aa3	AA+ AA AA-	AA+ AA AA-	aa+ aa aa-	An obligor has VERY STRONG capacity to meet its financial commitments. It differs from the highest rated obligors only in small degree.
A1 A2 A3	A+ A A-	A+ A A-	a+ a a-	An obligor has STRONG capacity to meet its financial commitments but is somewhat more susceptible to the adverse effects of changes in circumstances and economic conditions than obligors in higher-rated categories.
Baa1 Baa2 Baa3	BBB+ BBB BBB-	BBB+ BBB BBB-	bbb+ bbb bbb-	An obligor has ADEQUATE capacity to meet its financial commitments. However, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity of the obligor to meet its financial commitments.
Ba1 Ba2 Ba3	BB+ BB BB-	BB+ BB BB-	bb+ bb bb-	An obligor is LESS VULNERABLE in the near term than other lower-rated obligors. However, it faces major ongoing uncertainties and exposure to adverse business, financial, or economic conditions which could lead to the obligor's inadequate capacity to meet its financial commitments.
B1 B2 B3	B+ B B-	B+ B B-	b+ b b-	An obligor is MORE VULNERABLE than the obligors rated 'BB', but the obligor currently has the capacity to meet its financial commitments. Adverse business, financial, or economic conditions will likely impair the obligor's capacity or willingness to meet its financial commitments.
Caa	CCC	CCC	ccc	An obligor is CURRENTLY VULNERABLE, and is dependent upon favourable business, financial, and economic conditions to meet its financial commitments.
Ca	CC	CC	cc	An obligor is CURRENTLY HIGHLY-VULNERABLE.
	C	C	c	The obligor is CURRENTLY HIGHLY-VULNERABLE to nonpayment. May be used where a bankruptcy petition has been filed.
C	D	D	d	An obligor has failed to pay one or more of its financial obligations (rated or unrated) when it became due.
e, p	pr	Expected		Preliminary ratings may be assigned to obligations pending receipt of final documentation and legal opinions. The final rating may differ from the preliminary rating.
WR				Rating withdrawn for reasons including: debt maturity, calls, puts, conversions, etc., or business reasons (e.g. change in the size of a debt issue), or the issuer defaults.
unsolicited	unsolicited			This rating was initiated by the ratings agency and not requested by the issuer.
	SD	RD		This rating is assigned when the agency believes that the obligor has selectively defaulted on a specific issue or class of obligations but it will continue to meet its payment obligations on other issues or classes of obligations in a timely manner.
NR	NR	NR		No rating has been requested, or there is insufficient information on which to base a rating.

— Investment grade —

— "Junk" or sub-investment grade —

§3 Intensity-Based Credit Risk Modelling

S&P Credit Ratings on Sovereign Debt (Sep. 2010)



Issuer	Local Currency	Foreign Currency
Argentina	B	B
Australia	AAA	AAA
Belgium	AA+	AA+
Brazil	BBB+	BBB–
Canada	AAA	AAA
China	A+	A+
France	AAA	AAA
Germany	AAA	AAA
Hong Kong	AA+	AA+
India	BBB–	BBB–
Italy	A+	A+
Japan	AA	AA
Mexico	A	BBB
Netherlands	AAA	AAA
Russia	BBB+	BBB
South Africa	A+	BBB+
South Korea	A+	A
Spain	AA	AA
Switzerland	AAA	AAA
Taiwan	AA–	AA–
Thailand	A–	BBB+
Turkey	BB+	BB
United Kingdom	AAA	AAA
United States	AAA	AAA

Local currency debt usually has lower credit risk than foreign currency debt, as local currency debt is backed by taxation power of the government.

§3 Intensity-Based Credit Risk Modelling

Discrete-time Case: Estimating Default Probabilities from Historical Default Data

- Historical default data provided by rating agencies can be used to estimate the probability of default (PD).

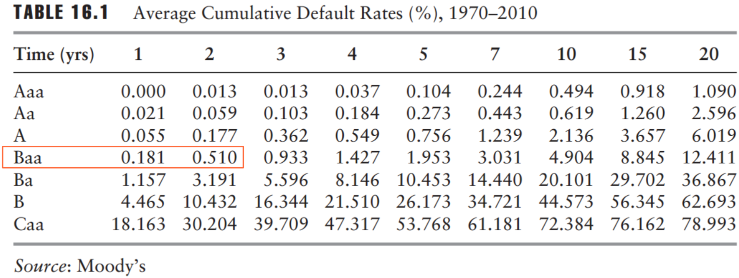


TABLE 16.1 Average Cumulative Default Rates (%), 1970–2010

Time (yrs)	1	2	3	4	5	7	10	15	20
Aaa	0.000	0.013	0.013	0.037	0.104	0.244	0.494	0.918	1.090
Aa	0.021	0.059	0.103	0.184	0.273	0.443	0.619	1.260	2.596
A	0.055	0.177	0.362	0.549	0.756	1.239	2.136	3.657	6.019
Baa	0.181	0.510	0.933	1.427	1.953	3.031	4.904	8.845	12.411
Ba	1.157	3.191	5.596	8.146	10.453	14.440	20.101	29.702	36.867
B	4.465	10.432	16.344	21.510	26.173	34.721	44.573	56.345	62.693
Caa	18.163	30.204	39.709	47.317	53.768	61.181	72.384	76.162	78.993

Source: Moody's

  - It shows PD for companies starting with a specified credit rating, e.g. a company with an initial credit rating of Baa has a probability of 0.181% of defaulting by the end of 1^{st} year, 0.510% by the end of 2^{nd} year, and so on.
  - PD during a particular year can be calculated, e.g. probability that a bond initially rated Baa will default during 2^{nd} year is $0.510\% - 0.181\% = 0.329\%$ (unconditional annual default probability as seem at time 0).

§3 Intensity-Based Credit Risk Modelling

Discrete-time Case: Average Cumulative Default Rates (1970-2010, Moody's)

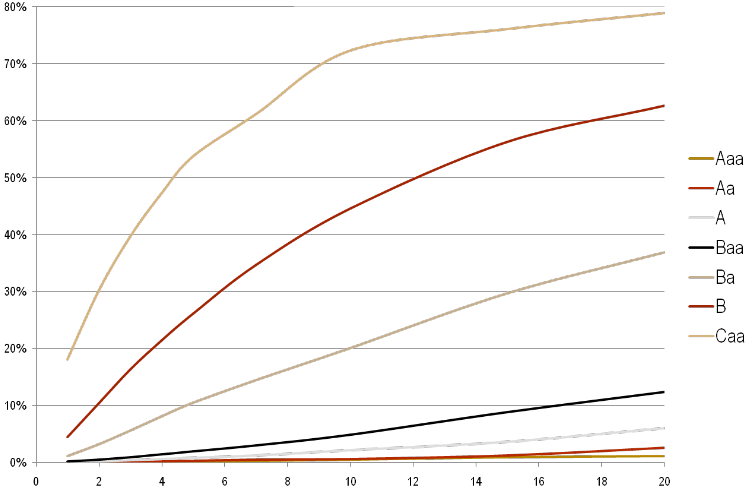


Figure: Average Cumulative Default Rates (1970-2010, Moody's)

§3 Intensity-Based Credit Risk Modelling

Discrete-time Case: Average Annual Default Rates (1970-2010, Moody's)

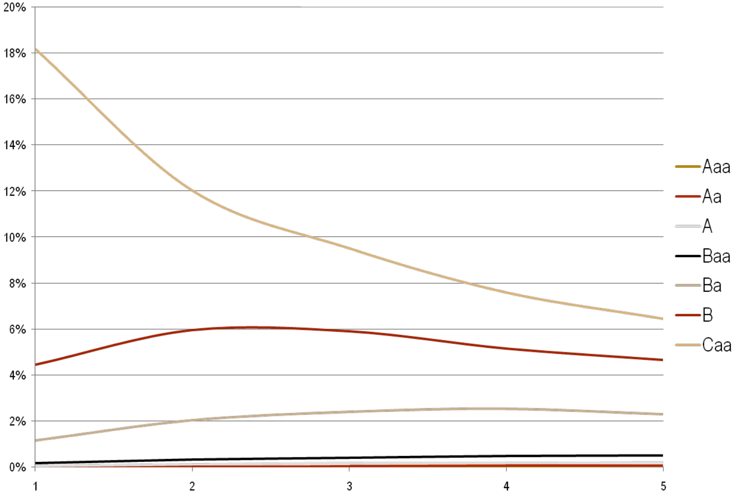
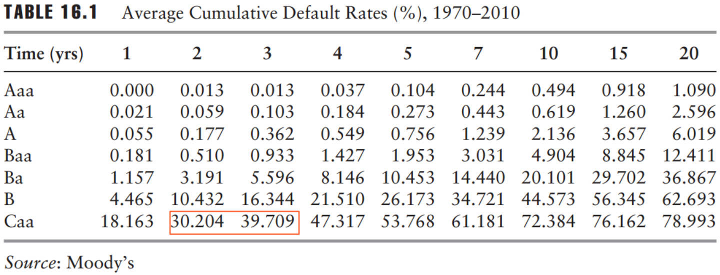


Figure: Average Annual Default Rates (1970-2010, Moody's)

§3 Intensity-Based Credit Risk Modelling

Discrete-time Case: Unconditional/Conditional Default Probabilities



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B	4.465	10.432	16.344	21.510	26.173	34.721	44.573	56.345	62.693
Caa	18.163	30.204	39.709	47.317	53.768	61.181	72.384	76.162	78.993

Source: Moody's

- The unconditional default probability is PD as seen at time 0.
  - e.g. probability of Caa bond defaulting during 3rd year is $39.709\% - 30.204\% = 9.505\%$.
- Probability that the Caa-rated bond will survive until the end of 2nd year is $1 - 30.204\% = 69.796\%$.
  - PD during 3rd year conditional on no earlier default is $9.505\%/69.796\% = 13.62\%$.

§3 Intensity-Based Credit Risk Modelling

Continuous-time Model: Default Intensity/Hazard Rate

- **Default intensity** (or *hazard rate*[1]) is PD over a short period given no earlier default, measuring instantaneous intensity of default (or bankruptcy, credit) events.

- Denote $\lambda(t)$ as the default intensity at time t, then, PD between times t and $t + \Delta t$, as seen at time t, conditional on no earlier default within time $[0, t]$, is approximately $\lambda(t)\Delta t$, i.e.

$$\lambda(t) := \lim_{\Delta t \to 0} \frac{\Pr\left\{t < \tau^* \leq t + \Delta t \mid \tau^* \geq t\right\}}{\Delta t}, \quad t \geq 0, \tag{3}$$

  where τ^* is a random default time (totally unpredictable, complete surprise: *stopping time*).

- We have the approximation

$$\Pr\left\{t \leq \tau^* \leq t + \Delta t \mid \tau^* \geq t\right\} \approx \lambda_t \Delta t. \tag{4}$$

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[1] *Hazard rate* is more general than *default intensity*. When the information filtration is only about default time, then, they are equivalent (Duffie, 2011, p.14).

- The cumulative survival probability by time t is given by

$$\Pr\{\tau^* > t\} = e^{-\int_0^t \lambda(s) \mathrm{d}s}. \tag{5}$$

- The cumulative default probability by time t is given by

$$F_{\tau^*}(t) := \Pr\{\tau^* \leq t\} = 1 - e^{-\int_0^t \lambda(s) \mathrm{d}s}. \tag{6}$$

- The default arrival is an *inhomogeneous Poisson process* of rate λ_t.

§3 Intensity-Based Credit Risk Modelling

Continuous-time Model: Default Intensity/Hazard Rate

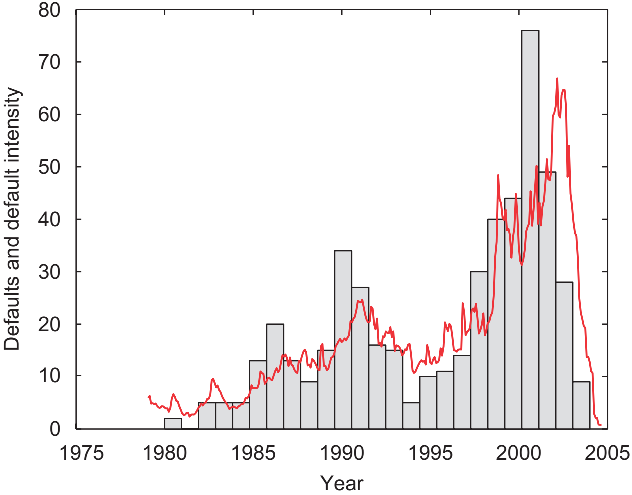


Figure: Total across firms of estimated default intensities (line), and the number of defaults in each year (bars), 1980–2004 (Duffie et al., 2007, p.651)

§3 Intensity-Based Credit Risk Modelling

Continuous-time Model: Default Intensity/Hazard Rate v.s. Default Probability Density

- The *default probability density* (Hull and White, 2000) is given by

$$f_{\tau^*}(t) := \frac{\mathrm{d}}{\mathrm{d}t} F_{\tau^*}(t) = \lambda(t) e^{-\int_0^t \lambda(s)\mathrm{d}s}, \tag{7}$$

  which means $f_{\tau^*}(t)\Delta t$ is approximately the unconditional PD between times t and $t + \Delta t$ as seen at time 0, i.e.

$$\Pr\left\{t \leq \tau^* \leq t + \Delta t\right\} \approx f_{\tau^*}(t)\Delta t; \tag{8}$$

  and links to *hazard rate* via

$$\lambda(t) = \frac{f_{\tau^*}(t)}{1 - F_{\tau^*}(t)}. \tag{9}$$

§3 Intensity-Based Credit Risk Modelling

Continuous-time Model: Default Intensity/Hazard Rate v.s. Default Probability Density

- Average *hazard rate* within time period $[t, T]$ is defined by

$$\overline{\lambda}_{[t,T]} := \frac{\int_t^T \lambda(s)\mathrm{d}s}{T-t}, \tag{10}$$

  then,

$$\Pr\{\tau^* \leq t\} = 1 - e^{-\overline{\lambda}_{[0,t]} \times t}. \tag{11}$$

- For constant *hazard rate*, i.e. $\lambda(t) \equiv \lambda$, then,

$$\Pr\{\tau^* \leq t\} = 1 - e^{-\lambda t}.$$

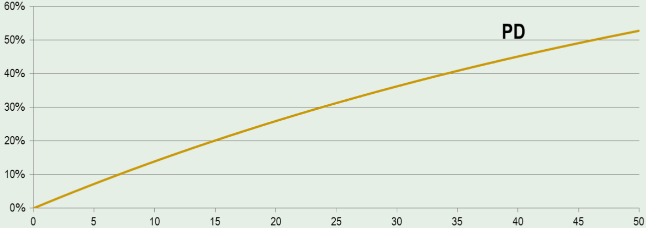
§3 Intensity-Based Credit Risk Modelling

Jarrow and Turnbull (1995) Model: Default Arrival of Homogenous Poisson Process

Example (Jarrow-Turnbull Model)

Suppose that the hazard rate $\lambda(t)$ is a constant 1.5% per year.

- PD by the end of 1^{st} year is $1 - e^{-0.015\times 1} = 1.49\%$.
- PD by the end of 2^{nd} year is $1 - e^{-0.015\times 2} = 2.96\%$.
- PD by the end of 3^{rd} , 4^{th} , 5^{th} years are similarly 4.40%, 5.82%, 7.23%.
- Unconditional PD during 4^{th} year is $5.82\% - 4.40\% = 1.42\%$.
- PD in 4^{th} year, conditional on no earlier default, is $1.42\%/(1 - 4.40\%) = 1.49\% \approx 1.5\%$.



Annual Defaults of Moody's-rated U.S. Firms, 1970–2008

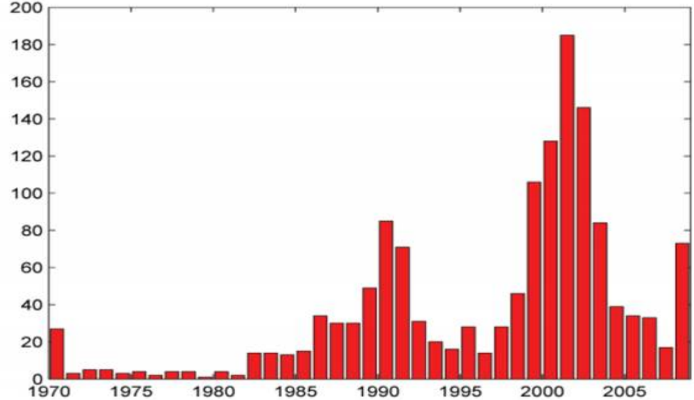
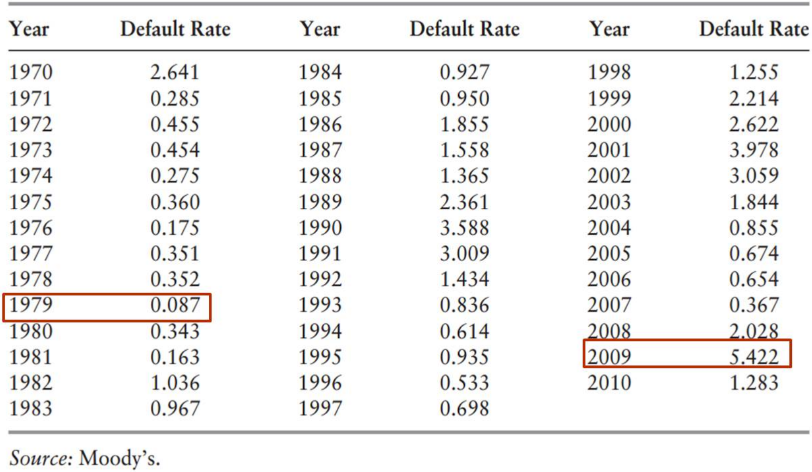


Figure: The peak in 1970 represents a cluster of 24 railway defaults triggered by the collapse of Penn Central Railway on June 21,1970. The fallout of the 1987 crash is indicated by the peak in the early 1990s. The burst of the internet bubble caused many defaults during 2001-2002. From a trough in 2007, default rates increased significantly in 2008. Source: Moody's Default Risk Service.

§3 Intensity-Based Credit Risk Modelling

Annual Percentage Default Rate (%) for All Rated Companies, 1970-2010



Year	Default Rate	Year	Default Rate	Year	Default Rate
1970	2.641	1984	0.927	1998	1.255
1971	0.285	1985	0.950	1999	2.214
1972	0.455	1986	1.855	2000	2.622
1973	0.454	1987	1.558	2001	3.978
1974	0.275	1988	1.365	2002	3.059
1975	0.360	1989	2.361	2003	1.844
1976	0.175	1990	3.588	2004	0.855
1977	0.351	1991	3.009	2005	0.674
1978	0.352	1992	1.434	2006	0.654
1979	0.087	1993	0.836	2007	0.367
1980	0.343	1994	0.614	2008	2.028
1981	0.163	1995	0.935	2009	5.422
1982	1.036	1996	0.533	2010	1.283
1983	0.967	1997	0.698		

Source: Moody's.

Cox (1955, 1972) Model: Defaults Arrival as A Doubly Stochastic Poisson Process

- Cox (1955, 1972) models, or *doubly stochastic Poisson processes*, are widely used for modelling event arrivals and survival analysis.
- They are based on conditional independence (doubly stochastic) assumption, i.e., default times follow independent Poisson processes given the intensities (Das et al., 2007, p.98).
- The *cumulative survival probability* by time t is

$$\Pr\{\tau^* > t\} = \mathbb{E}\left[e^{-\int_0^t \lambda(s)\mathrm{d}s}\right], \tag{12}$$

where the intensity λ_t is stochastic and independent of default.

§3 Intensity-Based Credit Risk Modelling

Recovery Rate

- The *recovery rate* of a bond is usually defined as the price of bond immediately (30 days) after default as a percentage (averagely 40%) of its face value.
  - Some claims have priorities over other claims and are met more fully, which depends on the bond holders' seniority.

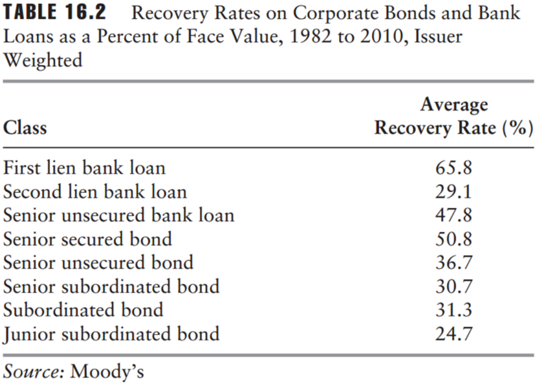


TABLE 16.2 Recovery Rates on Corporate Bonds and Bank Loans as a Percent of Face Value, 1982 to 2010, Issuer Weighted

Class	Average Recovery Rate (%)
First lien bank loan	65.8
Second lien bank loan	29.1
Senior unsecured bank loan	47.8
Senior secured bond	50.8
Senior unsecured bond	36.7
Senior subordinated bond	30.7
Subordinated bond	31.3
Junior subordinated bond	24.7

Source: Moody's

§3 Intensity-Based Credit Risk Modelling

Modeling Recovery Rates

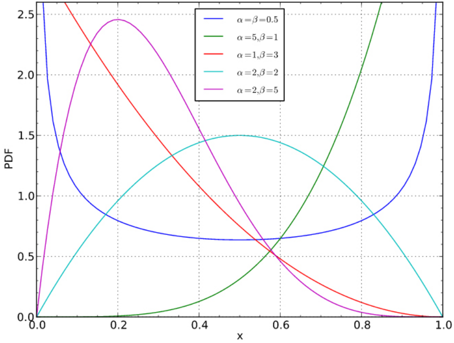
- Recovery rate can be modelled by *Beta distribution*, $X \sim \mathrm{Beta}(\alpha, \beta)$ with PDF

$$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in [0, 1], \tag{13}$$

where shape parameters $\alpha, \beta > 0$, and Γ is Gamma function, with mean and variance

$$m_X = \frac{\alpha}{\alpha+\beta}, \tag{14}$$

$$v_X = \frac{m_X(1-m_X)}{1+\alpha+\beta}. \tag{15}$$



- Recovery rates are significantly negatively correlated with default rates (Altman et al., 2005).
  - A bad year for default rate is usually doubly bad, because it is accompanied by a low recovery rate.
- Moody's best-fit estimate for 1982-2007 period is

$$\text{Average Recovery Rate} = 59.33 - 3.06 \times \text{ Non-investment Grade Default Rate}.$$

  - The correlation between the average recovery rate in a year and the non-investment grade default rate is about 50%.
- Jointly modelling for recovery rates and defaults rates based on shared covariates: Chava et al. (2011).

- Present value (at time 0) of *defaultable zero-coupon bond* which pays $\$1$ at maturity T is

$$v(0,T) = p(0,T)\Big(R + (1-R)\Pr\{\tau^* > T\}\Big),$$

  where
  - $p(0,T)$ is present value of *default-free zero-coupon bond* which pays $\$1$ at maturity T;
  - $R \in [0,1]$ is constant recovery rate (say, 40%) (which depends on seniority);
  - risk-free interest rate and default are assumed to be independent.

- *Risk premium* to compensate investors for taking default risk is

$$p(0,T) - v(0,T) = p(0,T)(1-R)\Pr\{\tau^* \le T\}. \tag{16}$$

- If $R = 0$ (no money is recovered if company defaults within period of $[0,T]$), then,

$$v(0,T) = p(0,T)\Pr\{\tau^* > T\}. \tag{17}$$

Credit Default Swaps (CDS)

- CDS was invented by economist Blythe Masters from JP Morgan in 1994.
- CDS buyer acquires protection or insurance from the seller against a credit event (i.e. default) by a particular company or country (i.e. reference entity).
- Premium is known as credit default spread (i.e. CDS spread), which is paid for life of contract or until default.
- CDS is a kind of insurance against credit (default) risk.

§3 Intensity-Based Credit Risk Modelling

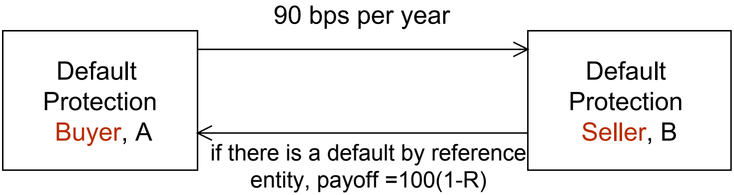
CDS' Global Market Size

§3 Intensity-Based Credit Risk Modelling

Example of CDS Cash-flow Structure

- *CDS buyer* A pays a premium (*credit spread*) of 90 bps per year on *face value* $N =$\$100 million to *CDS seller* B, for 5-year protection against default loss of *reference entity* X.

- If there is a default at time τ_X^*, $0 < \tau_X^* < 5$, CDS buyer A has the right to sell bonds with a face value of \$100 million issued by company X for \$100 million to *CDS seller* B.

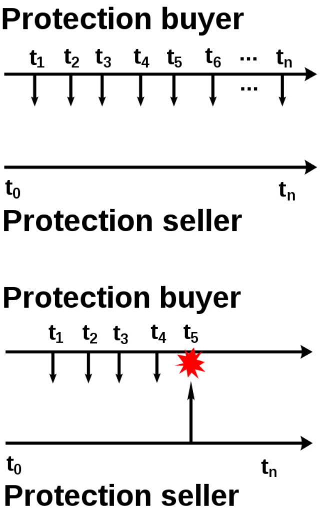


Recovery rate, *R*, is the ratio of bond value issued by reference entity X immediately after default to face value of bond.

§3 Intensity-Based Credit Risk Modelling

CDS Cash-flow Structure

- Protection buyer purchased a CDS at time t_0 and makes regular premium payments $N \times s_0$ at times $t_1, t_2, t_3, t_4 \dots$
- If reference entity suffers no credit event, then, buyer continues paying premiums until the end of contract at time $T = t_n$.
- If reference entity suffered a credit event, say, at $\tau_X^* = t_5$, then, protection seller pays buyer for the loss, and buyer would cease paying premiums to seller.



From *protection sellers*' point of view (ignoring *counterparty risk*):

- **Expected premium:**

$$Premium\ Leg = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} s_0 N e^{-rt_i}\mathbb{1}\{t_i < \tau_X^*\}\right], \tag{18}$$

  where s_0 is *CDS spread* at today $t = 0$ when CDS contract is created.

- **Expected loss:**

$$Loss\ Leg = \mathbb{E}^{\mathbb{Q}}\left[(1-R)Ne^{-r\tau_X^*}\mathbb{1}\{\tau_X^* \leq T\}\right]. \tag{19}$$

- Today $t = 0$, set CDS' PV = *expected premium* – *expected loss* =0, then,

$$s_0 = \frac{(1-R)\int_0^T e^{-ru} f_{\tau_X^*}(u)\mathrm{d}u}{\sum_{i=1}^n p(0, t_i)\,\mathrm{Pr}\{\tau_X^* > t_i\}}, \tag{20}$$

  where $f_{\tau_X^*}(u)$ is the density function of default time τ_X^*.

§3 Intensity-Based Credit Risk Modelling

CDS Protection Buyers and Sellers

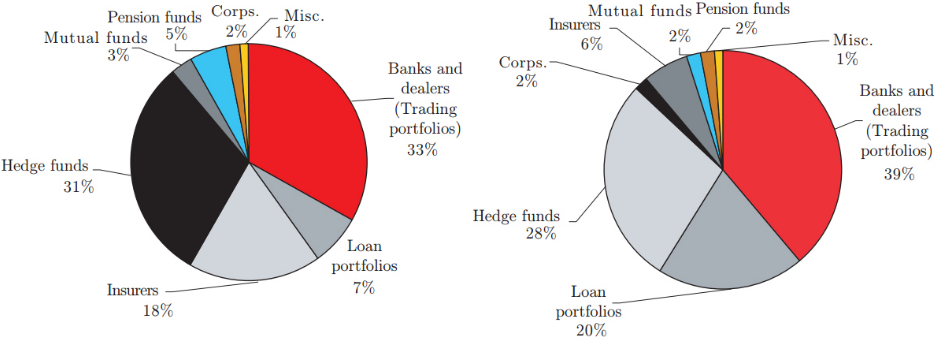
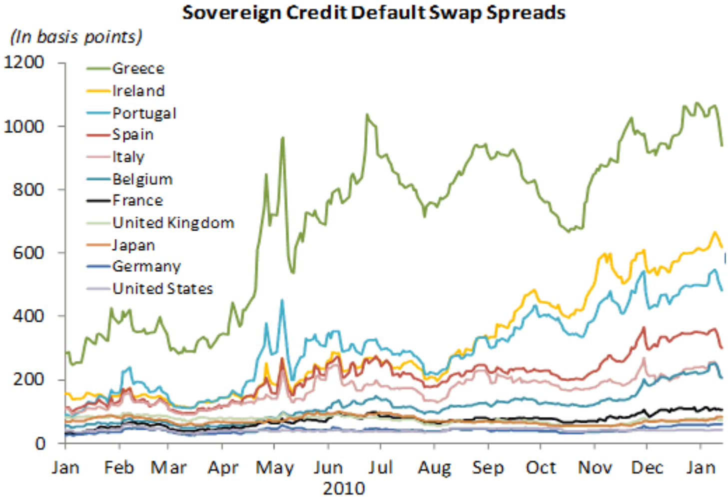


Figure: Estimated breakdown of CDS buyers (left) and sellers (right) of protection, Mar 2007 (Source: BoA)

§3 Intensity-Based Credit Risk Modelling

CDS Spread – “Fear Gauge" of Credit Risk



Source: Bloomberg L.P.

§3 Intensity-Based Credit Risk Modelling

CDS & Bond Yields

- *Credit spread* is the premium paid by protection buyer to seller, i.e. extra rate of interest per annum (quoted in *basis points*) required by investors for bearing credit risk, e.g.
    - *CDS spread*;
    - *Bond yield spread*, the amount by which the yield on a corporate bond exceeds the yield on a similar risk-free bond (e.g. US Treasury bond).

Example (Hedging & Arbitrage on Credit Risk)

A portfolio consists of a 5-year corporate bond yielding 7% per year and a long position in a 5-year CDS costing 200 bps (2%) per year, for the same reference entity.

- It is approximately a "risk-free" bond earning 5% per year, which is the implied risk-free rate in normal markets.
- In normal markets, what are arbitrage opportunities when real risk-free rate is 4.5%? what if it is 5.5%?

§3 Intensity-Based Credit Risk Modelling

iBoxx Bond Spread Indices 2006-2008

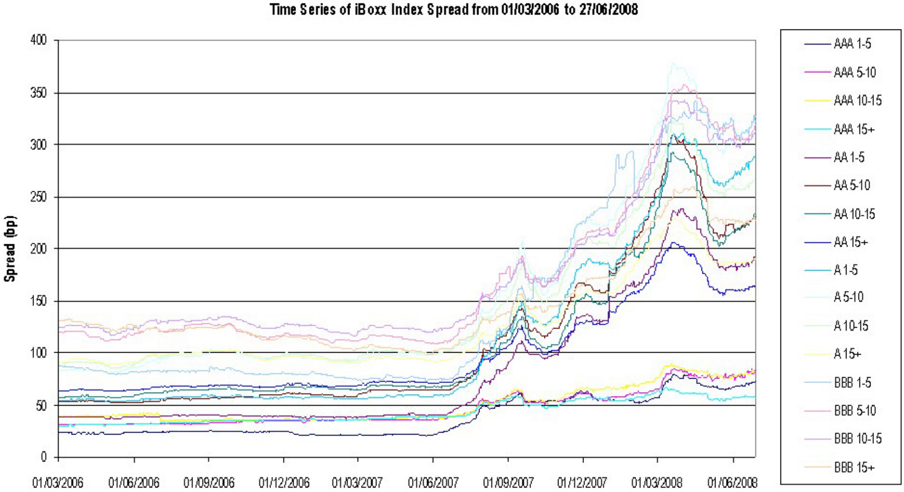


Figure: Bond Spread Indices 2006-2008 from Markit

§3 Intensity-Based Credit Risk Modelling

Default Intensity Implied from Credit Spreads

- Credit spread can be considered roughly to be a market's expected average loss rate (loss per time unit).
- Implied average risk-neutral default intensity over life of bond within time $[0, T]$ is approximately (Hull et al., 2005)

$$\overline{\lambda}_{[0,T]} = \frac{s(T)}{1-R}, \tag{21}$$

  where

  - $s(T)$ is (continuously compounding) credit yield spread over risk-free rate for a maturity of T, i.e.

$$s(T) = y(T) - r(T); \tag{22}$$

  - $0 \leq R < 1$ is recovery rate;
  - average hazard rate within time period $[t, T]$ is defined by

$$\overline{\lambda}_{[t,T]} := \frac{\int_t^T \lambda(s)\mathrm{d}s}{T-t}. \tag{23}$$

§3 Intensity-Based Credit Risk Modelling

Default Intensity Implied from Credit Spreads

- If credit spreads are known for a number of different maturities, term structure of hazard rate can be bootstrapped.

Example (Term Structure of Default Intensity Implied from CDS Spreads)

Suppose that CDS spread for 3-, 5-, and 10-year instruments is 50, 60, and 100 basis points, and expected recovery rate is 60%.

- Average hazard rate over 3 years is approximately $\bar{\lambda}_{[0,3]} = 0.005/(1-0.6) = 0.0125$.
- Average hazard rate over 5 years is approximately $\bar{\lambda}_{[0,5]} = 0.006/(1-0.6) = 0.015$.
- Average hazard rate over 10 years is approximately $\bar{\lambda}_{[0,10]} = 0.01/(1-0.6) = 0.025$.
- From this, we can estimate that the average hazard rate between year 3 and year 5 is $\bar{\lambda}_{[3,5]} = (5 \times 0.015 - 3 \times 0.0125)/2 = 0.01875$.
- Average hazard rate between year 5 and year 10 is $\bar{\lambda}_{[5,10]} = (10 \times 0.025 - 5 \times 0.015)/5 = 0.035$.

Real-World vs Risk-Neutral Default Probabilities

- Default probabilities backed out from bond prices or CDS spreads are risk-neutral default probabilities (conventionally denoted by $\mathbb{Q}$).
- Default probabilities backed out from historical default data are real-world (i.e. natural or physical) default probabilities (conventionally denoted by $\mathbb{P}$).
- For the same name and time to maturity, risk-neutral default probability are usually much higher than real-world default probability.
    - Difference between the two is particularly larger during crises due to investors' "flight to quality".

Real-World vs Risk-Neutral Default Probabilities

- Real-world default probabilities: calculate 7-year hazard rates from Moody's default data (1970-2010), Table 16.1.
- Risk-neutral default probabilities: estimate average 7-year hazard rates implied from bond prices of Merrill Lynch data (1996-2007).
- Assume risk-free rate equal to 7-year swap rate minus 10 bps, and recovery rate is 40%.

§3 Intensity-Based Credit Risk Modelling

Real-World vs Risk-Neutral Default Probabilities

Rating	Historical Hazard Rate (% per annum)	Hazard Rate from bonds (% per annum)	Ratio	Difference
Aaa	0.03	0.60	17.2	0.57
Aa	0.06	0.73	11.5	0.67
A	**0.18**	**1.15**	6.5	0.97
Baa	0.44	2.13	4.8	1.69
Ba	2.23	4.67	2.1	2.44
B	6.09	8.02	1.3	1.93
Caa	13.52	18.39	1.4	4.87

$$-\frac{1}{7}\ln(1-\boxed{0.01239}) = 0.0018 \qquad \frac{\boxed{0.05995} - 0.05308}{1-0.4} = 0.0115$$

$$\bar{\lambda}_{[0,T]} = -\frac{\ln(1-P\{\tau^* \le T\})}{T} \qquad \bar{\lambda}_{[0,T]} = \frac{y(T)-r(T)}{1-R} \qquad T \equiv 7, R \equiv 40\%$$

$$P\{\tau^* \le T\} = 1-e^{-\bar{\lambda}_{[0,T]}\times T} \qquad \bar{\lambda}_{[0,T]} = \frac{s(T)}{1-R}$$

Moody's default data *Merrill Lynch bond-price data*

- The ratio of the hazard rate backed out of bond prices to the hazard rate calculated from historical data is high for investment grade bonds, and tends to decline as the credit quality declines

§3 Intensity-Based Credit Risk Modelling

Risk Premiums Earned By Bond Traders

Rating	Bond Yield Spread over Treasuries (bps)	Spread of risk-free rate used by market over Treasuries (bps)	Spread to compensate for default rate in the real world (bps)	**Extra Risk Premium** (bps)
Aaa	78	42	2	34
Aa	86	42	4	40
A	**111**	**42**	**11**	**58**
Baa	169	42	26	101
Ba	322	42	132	148
B	523	42	355	126
Caa	1146	42	759	345

Expected 1-year default loss (real-world probability) = 1-year probability of default (calculated from the historical hazard rate from Moody's ,Table 16.1) multiplied by (1-R) where recovery rate R=0.4

Why Extra Risk Premium Exists?

- Corporate bonds are relatively illiquid and need additional compensation.
- Subjective default probabilities of bond traders may be much higher than the estimates from Moody's historical data.
- Bonds do not default independently of each other, which leads to systematic risk that cannot be diversified away; so bond traders require an excess expected return for bearing this risk.

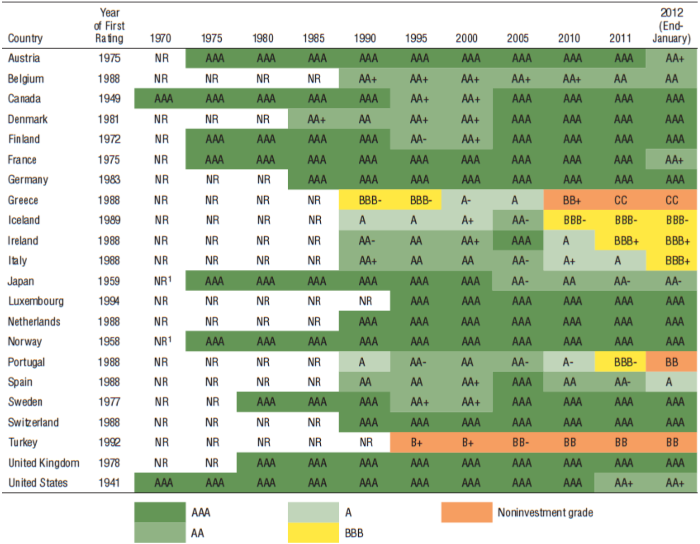
Which World Should We Use?

- We should use risk-neutral estimates for asset pricing, e.g. valuing credit derivatives and estimating the present value of default cost.
- We should use real-world estimates for risk management, e.g. calculating VaR and scenario analysis.

§4 Rating-Based Credit Risk Modelling

Historical Credit Rating Transition

History of S&P Sovereign Debt Credit Ratings by Country



Country	Year of First Rating	1970	1975	1980	1985	1990	1995	2000	2005	2010	2011	2012 (End-January)
Austria	1975	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AA+
Belgium	1988	NR	NR	NR	NR	AA+	AA+	AA+	AA+	AA+	AA	AA
Canada	1949	AAA	AAA	AAA	AAA	AAA	AA+	AA+	AAA	AAA	AAA	AAA
Denmark	1981	NR	NR	NR	AA+	AA	AA+	AA+	AAA	AAA	AAA	AAA
Finland	1972	NR	AAA	AAA	AAA	AAA	AA-	AA+	AAA	AAA	AAA	AAA
France	1975	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AA+
Germany	1983	NR	NR	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA
Greece	1988	NR	NR	NR	NR	BBB-	BBB-	A-	A	BB+	CC	CC
Iceland	1989	NR	NR	NR	NR	A	A	A+	AA-	BBB-	BBB-	BBB-
Ireland	1988	NR	NR	NR	NR	AA-	AA	AA+	AAA	A	BBB+	BBB+
Italy	1988	NR	NR	NR	NR	AA+	AA	AA	AA-	A+	A	BBB+
Japan	1959	NR[1]	AAA	AAA	AAA	AAA	AAA	AAA	AA-	AA	AA-	AA-
Luxembourg	1994	NR	NR	NR	NR	NR	AAA	AAA	AAA	AAA	AAA	AAA
Netherlands	1988	NR	NR	NR	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA
Norway	1958	NR[1]	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA
Portugal	1988	NR	NR	NR	NR	A	AA-	AA	AA-	A-	BBB-	BB
Spain	1988	NR	NR	NR	NR	AA	AA	AA+	AAA	AA	AA-	A
Sweden	1977	NR	NR	AAA	AAA	AAA	AA+	AA+	AAA	AAA	AAA	AAA
Switzerland	1988	NR	NR	NR	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA
Turkey	1992	NR	NR	NR	NR	NR	B+	B+	BB-	BB	BB	BB
United Kingdom	1978	NR	NR	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA
United States	1941	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AAA	AA+	AA+

🟩 AAA
🟩 AA
🟩 A
🟨 BBB
🟧 Noninvestment grade

Source: IMF (April 2012 "Global Financial Stability Report"

Jarrow-Lando-Turnbull Model: Dynamics of Credit Rating Transition

- **Jarrow-Lando-Turnbull model** (Jarrow et al., 1997): the dynamics of *credit rating transitions* is represented by a *discrete-time Markov chain*[2].
- To describe the dynamics of credit ratings quantitatively, let $\{X_t\}_{t=0,1,2,\ldots}$ represent (random) credit rating of a bond at time t, where X_t is a *time-homogeneous* *discrete-time Markov chain* on finite discrete-state space

  $$\mathcal{S} = \{1, 2, \ldots, K, K + 1\},$$

  where
  - 1^{st} state 1 represents the highest credit rating (e.g. AAA in S&P rankings);
  - K^{th} state K represents the lowest credit rating (e.g. C in S&P rankings);
  - the last state $K + 1$ represents default or bankruptcy, i.e. absorbing state which means once default, it will stay in the state of default forever;
  - to be consistent in notation, state 0 (excluded from $\mathcal{S}$ here) represents default-free.
- Default is modelled as the first time of this discrete-time Markov chain that hits the absorbing state (default state) $K + 1$.

---

[2] Similar idea was also adopted by Google co-founder Larry Page for his *PageRank*, Google's most well-known search ranking algorithm.

- $(K+1) \times (K+1)$ *time-homogeneous one-step transition matrix* is

$$\mathbb{Q}(t, t+1) \equiv \mathbb{Q} := \begin{pmatrix} q_{1,1} & \cdots & q_{1,k} & q_{1,K+1} \\ \vdots & \ddots & \vdots & \vdots \\ q_{K,1} & \cdots & q_{K,K} & q_{K,K+1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{24}$$

where

$$q_{i,j}(t, t+1) := \Pr\{X_{t+1} = j \mid X_t = i\} \equiv q_{i,j}, \qquad i, j \in \mathcal{S}, \quad \forall t = 0, 1, 2, \ldots;$$

$$q_{i,j} \in [0, 1], \quad \forall i \neq j, \qquad q_{i,i} = 1 - \sum_{j=1, j\neq i}^{K+1} q_{i,j}, \quad \forall i;$$

are actual (or natural) *transition probabilities* in one unit time (say, 1 year), and *absorbing state* for default is in the last row.

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Estimating the Natural-Probability Transition Matrix from Real Data

Credit Rating	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**	**NR**
AAA	87.44	7.37	0.46	0.09	0.06	0	0	0	4.59
AA	0.6	86.65	7.78	0.58	0.06	0.11	0.02	0.01	4.21
A	0.05	2.05	86.96	5.5	0.43	0.16	0.03	0.04	4.79
BBB	0.02	0.21	3.85	84.13	4.39	0.77	0.19	0.29	6.14
BB	0.04	0.08	0.33	5.27	75.73	7.36	0.94	1.2	9.06
B	0	0.07	0.2	0.28	5.21	72.95	4.23	5.71	11.36
CCC/C	0.08	0	0.31	0.39	1.31	9.74	46.83	28.83	12.52

Figure: *Global Average One-Year Transition Rates* (%), 1981–2004, Source: Standard & Poor

- Estimate the transition matrix $\mathbb{Q}$ by eliminating Not-rated (NR) data:

Rating	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**
AAA	0.916369734	0.077237476	0.004820792	0.000943198	0.000628799	0	0	0
AA	0.006262394	0.904394113	0.08120238	0.006053648	0.000626239	0.001148106	0.000208746	0.000104373
A	0.0005251	0.021529091	0.913253518	0.057760975	0.004515858	0.001680319	0.00031506	0.00042008
BBB	0.000213106	0.002237613	0.041022909	0.896430474	0.046776771	0.008204582	0.002024507	0.003090037
BB	0.000439802	0.000879604	0.003628367	0.057943925	0.832655305	0.080923584	0.010335349	0.013194063
B	0	0.000789622	0.002256063	0.003158488	0.058770446	0.822899041	0.047715736	0.064410603
CCC/C	0.00091439	0	0.003543262	0.004457652	0.01497314	0.111327009	0.535261173	0.329523374
D	0	0	0	0	0	0	0	1

Figure: Estimated Transition Matrix $\mathbb{Q}$

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Term Structure of the Natural Default Probability

- τ_i^* is denoted as default time (absorption state) of X_t with the current credit rating $X_0 = i$, i.e.

$$\tau_i^* := \inf\{t \geq 0 : X_0 = i, X_t = K + 1\}. \tag{25}$$

- Natural default probability within time T for the current i-rated bonds is

$$\Pr\{\tau_i^* \leq T\} = q_{i,K+1}(0, T), \tag{26}$$

where $q_{i,K+1}(0, T)$ is from the T*-step transition matrix*

$$\mathbb{Q}(0, T) = \mathbb{Q}^{(T)} = \mathbb{Q}^T. \tag{27}$$

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Term Structure of the Natural Default Probability

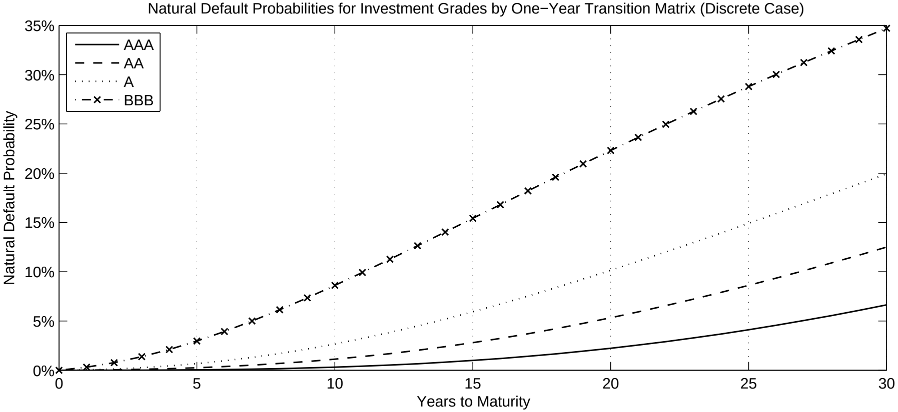


Figure: 30-Year Term Structure of Natural Default Probabilities for Investment Grades

Jarrow-Lando-Turnbull Model: Credit Rating Transition under the Risk-Neutral Probability

- For the rating process after risk neutralization, $\tilde{X}_t$, assume its associated $(K+1)\times(K+1)$ *one-step transition matrix* is now time-non-homogeneous, i.e.

$$\tilde{\mathbb{Q}}(t,t+1) := \begin{pmatrix} \tilde{q}_{1,1}(t,t+1) & \cdots & \tilde{q}_{1,K}(t,t+1) & \tilde{q}_{1,K+1}(t,t+1) \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{q}_{K,1}(t,t+1) & \cdots & \tilde{q}_{K,K}(t,t+1) & \tilde{q}_{K,K+1}(t,t+1) \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

  where

$$\tilde{q}_{i,j}(t,t+1) := \widetilde{\Pr}\left\{X_{t+1}=j \mid X_t=i\right\}, \quad i,j\in\mathcal{S}, \quad \forall t=0,1,2,...; \tag{28}$$

$$\tilde{q}_{i,j}(t,t+1)\in[0,1], \quad \forall i\neq j, \qquad \tilde{q}_{i,i}(t,t+1) = 1-\sum_{j=1,j\neq i}^{K+1}\tilde{q}_{i,j}(t,t+1), \quad \forall i;$$

  are risk-neutral *transition probabilities* in one unit time.

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Credit Rating Transition under the Risk-Neutral Probability

- Assume risk-neutral transition probabilities can be transferred from the corresponding actual *transitional probabilities* by

$$\tilde{q}_{i,j}(t, t+1) := \pi_{i,j}(t, t+1) q_{i,j}, \quad \forall j \neq i, \tag{29}$$

  where

  - $q_{i,j}$ is actual *transitional probabilities* of the observed *time-homogeneous Markov chain* X_t;
  - $\pi_{i,j}(t, t+1)$ are *risk premium adjustments*.

- For simplicity, further assume

$$\pi_{i,j}(t, t+1) = \pi_i(t, t+1), \quad \forall j \neq i, \tag{30}$$

  which are deterministic functions of time t such that $\tilde{q}_{i,j}(t, t+1) \in [0, 1]$ for all i, j.

- $\tilde{X}_t$ and spot risk-free interest rate process are assumed to be mutually independent under risk-neutral measure.

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Risk-Neutral Default Probability

- $\tilde{\tau}_i^*$ is denoted as default time (absorption state) of $\tilde{X}$ when $\tilde{X}_0 = i$, i.e.

$$\tilde{\tau}_i^* := \inf\{t \geq 0 : \tilde{X}_0 = i, \tilde{X}_t = K + 1\}. \tag{31}$$

- Risk-adjusted survival probability is

$$\widetilde{\Pr}\{\tilde{\tau}_i^* > T\} = \sum_{k=1}^{K} \tilde{q}_{i,k}(0, T) = 1 - \tilde{q}_{i,K+1}(0, T), \tag{32}$$

where $\tilde{q}_{i,K+1}(0, T)$ can be obtained from time-non-homogeneous T*-step transition matrix*

$$\tilde{\mathbb{Q}}(0, T) = \tilde{\mathbb{Q}}(0, 1) * \tilde{\mathbb{Q}}(1, 1 + 1) * \ldots * \tilde{\mathbb{Q}}(T - 1, T). \tag{33}$$

- Present value of defaultable zero-coupon bond of i^{th}-class credit rating which needs pay \$ 1 at maturity T is

$$v_i(0,T) = v_0(0,T)\left(R + (1-R)\widetilde{\Pr}\{\tilde{\tau}_i^* > T\}\right), \qquad i = 1, 2, ..., K, \tag{34}$$

  where

  - $v_0(0,T)$ is present value of a default-free zero-coupon bond which pays $\$1$ at maturity T;
  - $R \in [0,1]$ is constant recovery rate (say, 40%);
  - risk-free interest rate and default are assumed to be independent.

Jarrow-Lando-Turnbull Model: Numerical Implementation

- Assume there are only 3 states of creditworthiness: I = Investment Grade, J = Junk Grade, D = Default (absorbing), with *one-year transition matrix*

$$\mathbb{Q} = \begin{array}{cc} \begin{pmatrix} 0.90 & 0.05 & 0.05 \\ 0.10 & 0.80 & 0.10 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} I \\ J \\ D \end{array} \end{array}.$$

- Given the associated risk-free interest rate and credit spreads by

$$\begin{pmatrix} r_{01} \\ r_{02} \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.09 \end{pmatrix}, \begin{pmatrix} s_{I,01} \\ s_{I,02} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.015 \end{pmatrix}, \begin{pmatrix} s_{J,01} \\ s_{J,02} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.03 \end{pmatrix}.$$

- Assume there is no correlation between credit rating migration and interest rate.

- Market traded prices of defaultable zero-coupon bonds of maturities $T = 1, 2$ for ratings I, J are observed as

$$B_I(0,1) = \frac{1}{1.09},\ B_I(0,2) = \frac{1}{1.105^2};\ B_J(0,1) = \frac{1}{1.10},\ B_J(0,2) = \frac{1}{1.12^2}.$$

§4 Rating-Based Credit Risk Modelling

Jarrow-Lando-Turnbull Model: Numerical Implementation

- Payoff vector is $C := \begin{pmatrix} 1 \\ 1 \\ R \end{pmatrix}$ where recovery rate R is assumed to be $R = 40\%$.
- If the current (at time $t = 0$) state is I, then, we transform natural probabilities Q_I into risk-neutral probabilities $\tilde{Q}_I$ by adjustment π_I,

$$Q_I = \begin{pmatrix} 0.90 \\ 0.05 \\ 0.05 \end{pmatrix} \longrightarrow \tilde{Q}_I = \begin{pmatrix} 1 - 0.10\pi_I(0,1) \\ 0.05\pi_I(0,1) \\ 0.05\pi_I(0,1) \end{pmatrix}.$$

- We can calibrate *risk-premium adjustment* π_I, by making the expected value of discounted cash-flows equal to the traded price of bond in market, i.e.

$$B_I(0,1) = \frac{1}{1+r_{01}} C^T \tilde{Q}_I \quad \text{i.e.} \quad \frac{1}{1.09} = \frac{1}{1.08} \begin{pmatrix} 1 & 1 & 0.4 \end{pmatrix} \begin{pmatrix} 1 - 0.10\pi_I(0,1) \\ 0.05\pi_I(0,1) \\ 0.05\pi_I(0,1) \end{pmatrix}$$

giving $\pi_I(0,1) = 0.30581$.

- Similarly, for calibrating π_J, we have

$$B_J(0,1) = \frac{1}{1+r_{01}} C^T \tilde{Q}_J \quad \text{i.e.} \quad \frac{1}{1.10} = \frac{1}{1.08} \begin{pmatrix} 1 & 1 & 0.4 \end{pmatrix} \begin{pmatrix} 0.10\pi_J(0,1) \\ 1 - 0.20\pi_J(0,1) \\ 0.10\pi_J(0,1) \end{pmatrix} \tag{35}$$

giving $\pi_J(0,1) = 0.30303$.

- Implied risk-neutral transition matrix within the first year is

$$\tilde{\mathbb{Q}}(0,1) = \begin{array}{c} \begin{pmatrix} 0.9694 & 0.0153 & 0.0153 \\ 0.0303 & 0.9394 & 0.0303 \\ 0 & 0 & 1.00 \end{pmatrix} \begin{array}{c} I \\ J \\ D \end{array} \end{array}. \tag{36}$$

§5 Equity-Based Credit Risk Modelling

Merton's Structure Model: A View from Corporate Finance

- Information about equity prices is more up-to-date than credit ratings.
- Merton's model (Merton, 1974) relates credit risk of a (limited-liability) firm to its capital structure (assets and liabilities), and regards equity as an option on firm value.
- Assumptions:
  1. Firm is funded by equity and debt, i.e.

     $$V_t = E_t + B_t, \qquad t \geq 0,$$

     where V_t is firm value (total value of firm's assets), E_t is equity value, B_t is debt value.
  2. Debt is a zero-coupon bond with a constant debt repayment D at maturity T.
  3. V_t under risk-neutral measure follows SDE

     $$\frac{\mathrm{d}V_t}{V_t} = r\mathrm{d}t + \sigma_V \mathrm{d}W_t,$$

     where σ_V is volatility of firm value.

- By capital structure and bankruptcy law:

Default State	Firm Value	Debt Value	Equity Value
no default	$V_T \geq D$	D	$V_T - D$
default	$V_T < D$	V_T	0

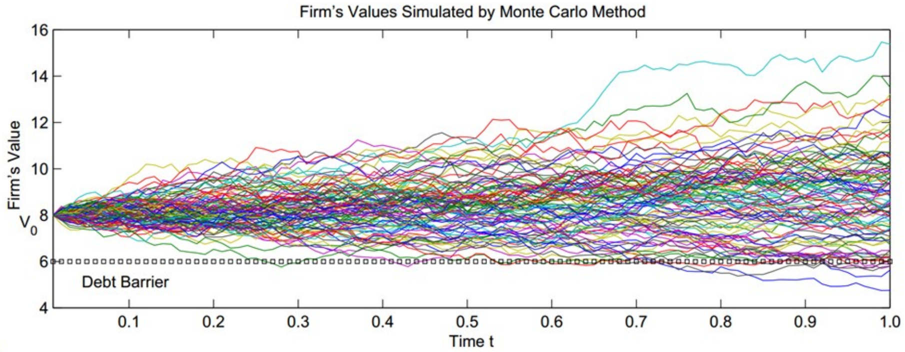
- Then, equity value E_T (i.e. payment to shareholders at time T) is

$$E_T = \max\left\{V_T - D, 0\right\}.$$

- Shareholders are long a call option on its asset value with strike D and maturity T; debtholders are short a put option with same strike and maturity.

§5 Equity-Based Credit Risk Modelling

Merton's Structure Model



$$r = 5\%, \sigma_V = 0.2, D = 6, V_0 = 8, N = 100$$

- By B-S formula, firm's equity price today is

$$E_0 = V_0\Phi(d_1) - De^{-rT}\Phi(d_2),$$

where

$$d_1 := \frac{\ln\frac{V_0}{D} + \left(r + \frac{\sigma_V^2}{2}\right)T}{\sigma_V\sqrt{T}},$$

$$d_2 := \frac{\ln\frac{V_0}{D} + \left(r - \frac{\sigma_V^2}{2}\right)T}{\sigma_V\sqrt{T}} = d_1 - \sigma_V\sqrt{T}.$$

- Risk-neutral PD is

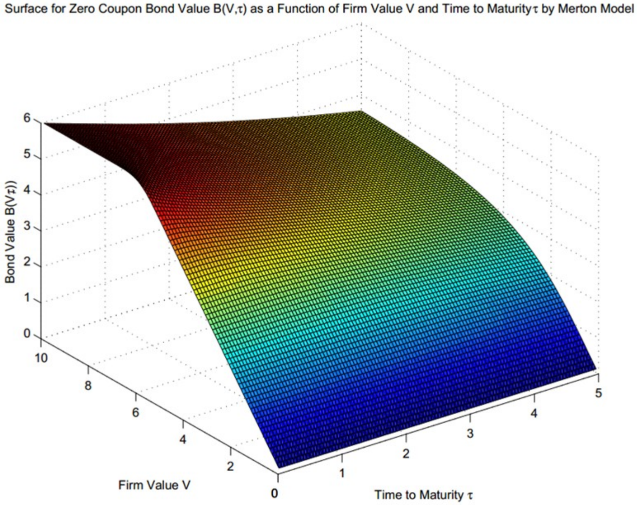
$$\Pr\{V_T \le D\} = \Phi(-d_2).$$

- Value of defaultable zero-coupon bond today is

$$B_0 = V_0 - E_0 = V_0\Phi(-d_1) + De^{-rT}\Phi(d_2).$$

§5 Equity-Based Credit Risk Modelling

Merton's Structure Model



$$r = 5\%, \sigma_V = 0.2, D = 6$$

§5 Equity-Based Credit Risk Modelling

Merton's Structure Model: Distance-to-Default (DtD)

- *Distance-to-Default* (DtD) is the number of standard deviations of firm's value that must change for default to be triggered T years in future, i.e.

$$\mathrm{DtD} := d_2 = \frac{\ln \frac{V_0}{D} + \left(r - \frac{\sigma_V^2}{2} \right) T}{\sigma_V \sqrt{T}}.$$

- The smaller the value of DtD, the larger the probability of default.

- DtD is essentially a volatility-corrected measure of leverage (Duffie et al., 2007, p.639), an important factor (accounting measure) for forecasting default (Bharath and Shumway, 2008).

Merton's Structure Model: Distance-to-Default (DtD)

- Estimated default intensities are strongly monotonically decreasing in DtD: a 10% reduction in distance to default causes an estimated 11.3% proportional increase in default intensity (Duffie et al., 2007, p.649).

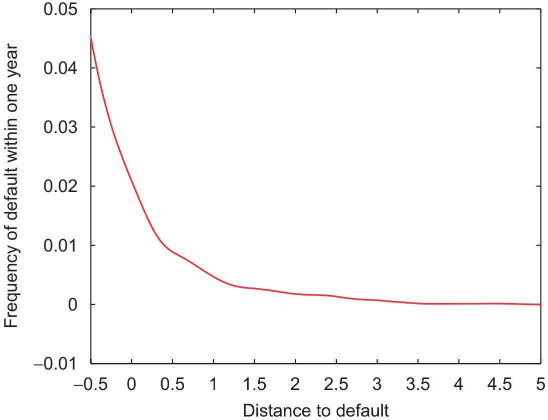


Figure: Empirical one-year default frequency as a function of DtD with kernel smoothing

§5 Equity-Based Credit Risk Modelling

Merton's Structure Model

- Firm's value V_t is unobservable, its initial value V_0 and volatility σ_V need calibration.

- By *Ito's Lemma*,

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0,$$

  where $\frac{\partial E}{\partial V} = \Phi(d_1)$, and volatility of equity price σ_E can be estimated (Jones et al., 1984).

- Two equations of (V_0, σ_V) enable V_0 and σ_V to be determined (implied) from E_0 and σ_E.

§5 Equity-Based Credit Risk Modelling

Numerical Example of Merton's Structure Model

Example (Merton's Structure Model)

A company's equity E_0 is \$3 million, volatility of equity σ_E is 80%. Risk-free rate r is 5%, debt D is \$10 million, time to maturity T is 1 year.

- Solving the two equations (via Excel 'Solver') gives $V_0 = 12.40$, $\sigma_V = 21.23\%$.
- 1-year PD is
$$PD_{T=1} = \Phi(-d_2) = 12.7\%.$$
- The current implied market value of debt (zero-coupon bond) is
$$B_0 = V_0 - E_0 = 12.4 - 3 = 9.40.$$
- Present value of promised payment is $10 \times e^{-5\% \times 1} = 9.51$.
- Expected loss percentage is
$$L\% = (9.51 - 9.40)/9.51 = 1.2\%.$$
- Recovery rate is $R = 91\%$, implied from equation $L\% = PD_{T=1} \times (1 - R)$.
- 1-year DtD $= 1.14$.

§5 Equity-Based Credit Risk Modelling

Industrial Applications of Merton’s Structure Model

1. KMV model
2. JP Morgan’s CreditMetrics (Gupton et al., 1997)
3. Basel 2, 2.5, 3
4. CreditMetrics’ CreditGrades (Finkelstein et al., 2002)

§5 Equity-Based Credit Risk Modelling

Imperfections of Merton's Structure Model

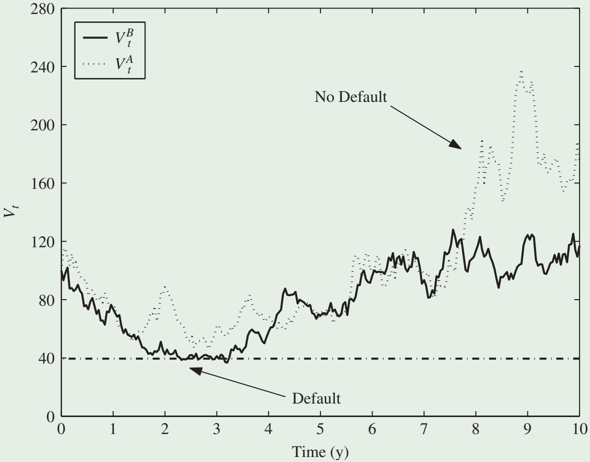
1. Default can only occur at maturity T, no matter the behaviour of asset value before T.

2. Capital structure is too simple: e.g. debt is a simple zero-couple bond.

3. Default can be predicted with increasing precision as time passes, which is due to the path continuity of geometric Brownian motion.

Example (Black-Cox Model)

- Black and Cox (1976) allows default time be any time within $(0, T]$:



§5 Equity-Based Credit Risk Modelling

Extensions of Merton's Structure Model: First-passage Time Models

Example (Black-Cox Model)

- Default time is defined by

$$\tau^* := \inf \left\{ t > 0 \mid V_t \leq D \right\},$$

  i.e. the first-passage time hitting the continuously-monitored *default barrier* $D < S_0$.

- Cumulative survival probability by time T at time 0 is

$$\Pr\{\tau^* > T\} = \Phi(d_1) - \left(\frac{D}{V_0} \right)^{\frac{2r}{\sigma_V^2} - 1} \Phi(d_2).$$

- Other first-passage time models: time-dependent barrier of Black and Cox (1976), stochastic barrier of Kim et al. (1993), see also Fischer et al. (1989); Leland (1994).

Correlation of Returns

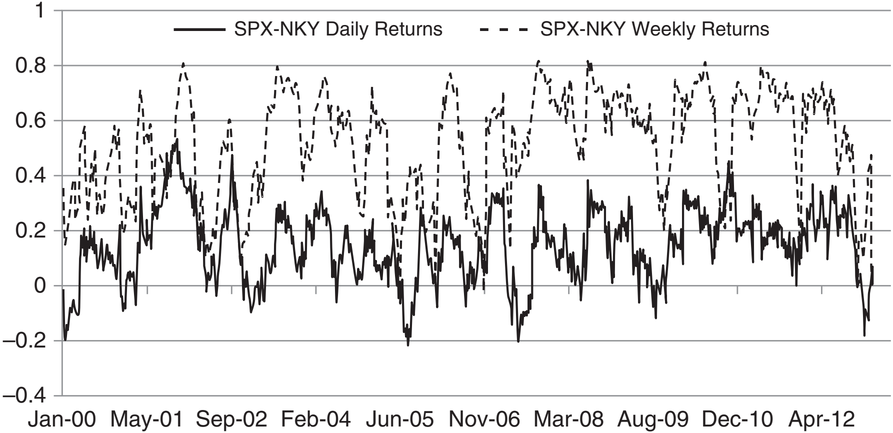


Figure: Historical correlation of daily and weekly returns between S&P500 and Nikkei225 over a 3-month rolling window since 2000

§6 Modelling Dependent Defaults

Dependent Defaults via Correlation

- Given a time horizon T, *default-event correlation* between two names is the correlation between *default indicators* $\mathbb{1}\{\tau_1^* < T\}$ and $\mathbb{1}\{\tau_2^* < T\}$, i.e., *Pearson correlation coefficient*

$$\begin{aligned}\rho_{1,2}(T) \quad &:= \quad \frac{\mathbb{E}\left[\mathbb{1}\{\tau_1^* < T\}\mathbb{1}\{\tau_2^* < T\}\right] - \mathbb{E}\left[\mathbb{1}\{\tau_1^* < T\}\right]\mathbb{E}\left[\mathbb{1}\{\tau_2^* < T\}\right]}{\sqrt{\left(\mathbb{E}\left[\mathbb{1}\{\tau_1^* < T\}^2\right] - \mathbb{E}\left[\mathbb{1}\{\tau_1^* < T\}\right]^2\right)\left(\mathbb{E}\left[\mathbb{1}\{\tau_2^* < T\}^2\right] - \mathbb{E}\left[\mathbb{1}\{\tau_2^* < T\}\right]^2\right)}} \\ &= \quad \frac{p_{1,2}(T) - p_1(T)p_2(T)}{\sqrt{p_1(T)\Big(1 - p_1(T)\Big)p_2(T)\Big(1 - p_2(T)\Big)}},\end{aligned}$$

where marginal default probabilities

$$p_1(T) := \mathbb{E}\Big[\mathbb{1}\{\tau_1^* \leq T\}\Big], \qquad p_2(T) := \mathbb{E}\Big[\mathbb{1}\{\tau_2^* \leq T\}\Big],$$

and joint default probability

$$p_{1,2}(T) := \mathbb{E}\Bigg[\mathbb{1}\{\tau_1^* < T\}\mathbb{1}\{\tau_2^* < T\}\Bigg].$$

- It only depends on the first two moments.
- For empirical analysis by S&P, see De Servigny and Renault (2002).

Examples of Bivariate Uniform Distribution $\mathcal{U}[0,1]^2$: Dependent But Zero Correlation

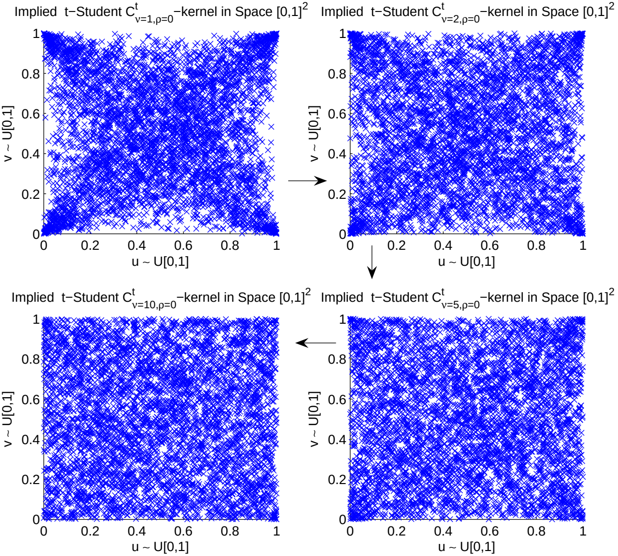
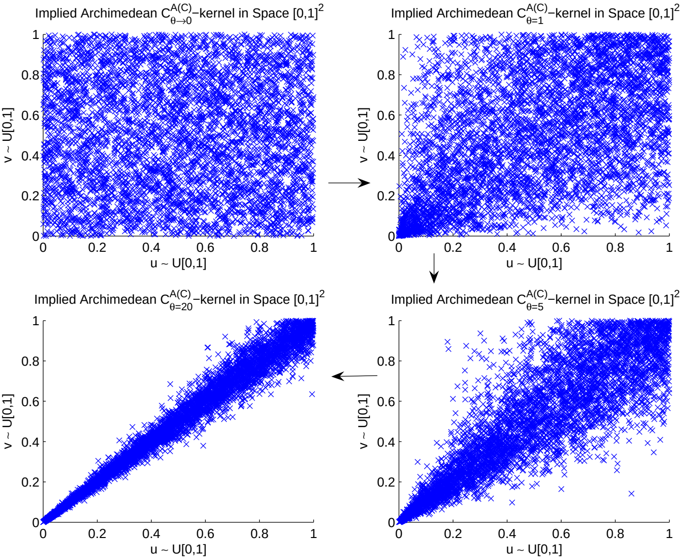


Figure: Zero-correlation Dependent Bivariate Uniform Distributions by *t*-student Copula of 5000 Samples

§6 Modelling Dependent Defaults

Examples of Bivariate Uniform Distribution $\mathcal{U}[0,1]^2$: Different Dependency Structures



Implied Archimedean $C^{A(C)}_{\theta\to 0}$−kernel in Space $[0,1]^2$

Implied Archimedean $C^{A(C)}_{\theta=1}$−kernel in Space $[0,1]^2$

Implied Archimedean $C^{A(C)}_{\theta=20}$−kernel in Space $[0,1]^2$

Implied Archimedean $C^{A(C)}_{\theta=5}$−kernel in Space $[0,1]^2$

Figure: Dependent Bivariate Uniform Distributions by Archimedean Copula of 5000 Samples

§6 Modelling Dependent Defaults

Examples of 3D Uniform Distribution $\mathcal{U}[0,1]^3$: Different Dependency Structures

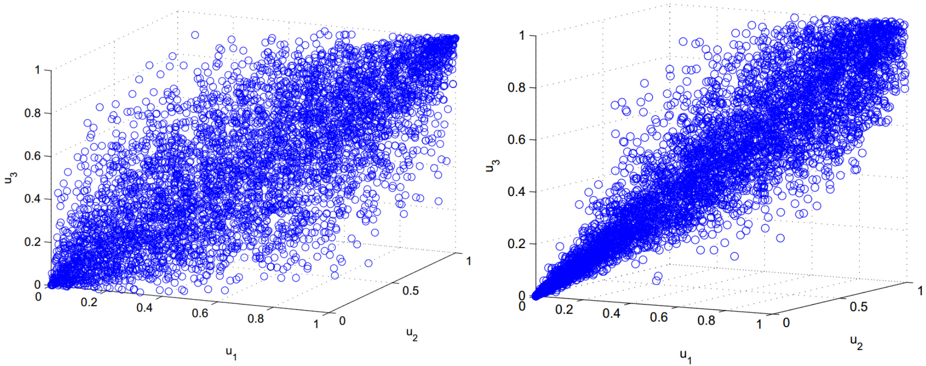


Figure: Gaussian v.s. Clayton Copulas: $(U_1, U_2, U_3) \sim \mathcal{U}[0,1]^3$ of Different Dependency with 5,000 Samples

Dependent Defaults via Copulas

- *Copula* is a *multivariate probability distribution* for which *marginal probability distribution* of each variable is uniform.
- Theoretical foundation, *Sklar's Theorem* (Sklar, 1959) states that, any *multivariate joint distribution* can be written in terms of *univariate marginal distribution functions* and a copula which describes the dependence structure between variables.
- Recommend books:
  1. *An Introduction to Copulas* (Nelsen, 2006)
  2. *Copula Methods in Finance* (Cherubini et al., 2004)

Dependent Defaults via Copulas: The Portfolio Loss Distribution

- For a portfolio of n defaultable bonds, denote τ_i^* as the default time of i^{th} bond, then, the total number of defaults within time period $[0, T]$ is
$$N_T = \sum_{i=1}^{n} \mathbb{1}\{\tau_i^* \leq T\},$$
where default times $\{\tau_i^*\}_{i=1,2,\ldots,n}$ could be dependent.

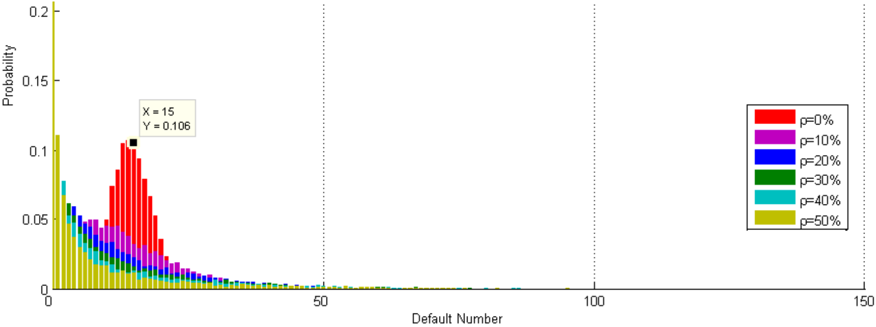


Figure: Distribution of default number for different ρ, homogenous $\lambda = 5\%$, $T = 1$, $n = 300$, 10000 samples

§7 Pricing Collateralized Debt Obligation (CDO)

Basis of Asset-Backed Securities

- *Structured finance* was initially developed by US banking world in early 1980s (in *mortgage-backed-securities* (MBS) format), in order to reduce *regulatory capital* requirements by removing and transferring risk from balance sheet to other parties[3].
- *Asset-backed securities* (ABS)[4] and MBS contracts are not yet standardized.
- However, there are certain features that emerge in virtually any ABS deal, the most important of which are
  1. default risk,
  2. loss-given-default (LGD), or recovery rate,
  3. prepayment risk (due to the amortization of principal value).
- Reality shows negative correlation between default and prepayment.

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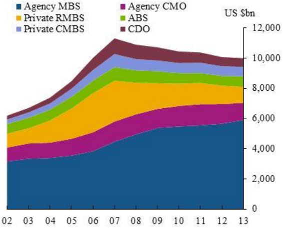
[3] However, some counter-examples have been found in Acharya et al. (2013) that the motivation of securitization (for *asset-backed commercial papers*) is not necessary to remove and transfer risk but to take more risk due via *implicit guarantees*.

[4] Workshop on ABS by Prof. Giddy at NYU: http://giddy.org/abs-hypo.htm

§7 Pricing Collateralized Debt Obligation (CDO)

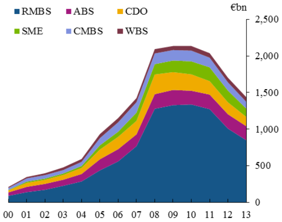
Basis of Asset-Backed Securities: Outstanding and Issuance of US/EU Securitisation

Chart 1: US securitisation outstanding



Sources: SIFMA.

Chart 2: European securitisation outstanding [(a)]



Sources: SIFMA and Bank calculations.

(a) Includes retained deals

Chart 3: US securitisation issuance

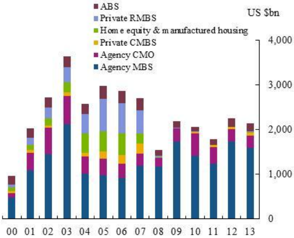
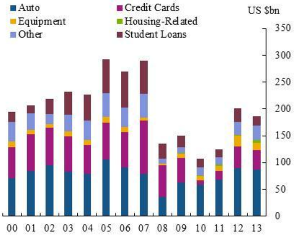


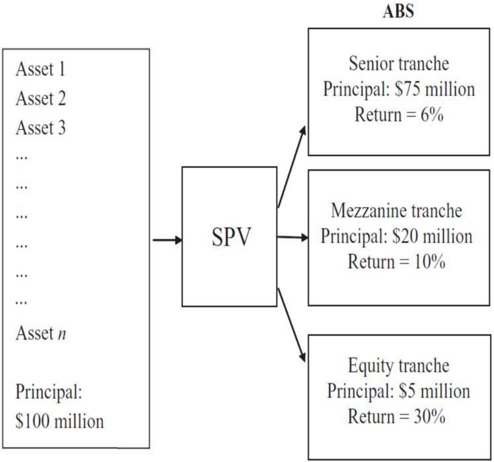
Chart 4: US ABS issuance



§7 Pricing Collateralized Debt Obligation (CDO)

Asset-Backed Security (ABS)

- ABS security is created from cash flows of financial assets (such as loans, bonds, credit card receivables, mortgages, auto loans).
- A portfolio of assets (such as subprime mortgages) is sold by the originators of assets to a *special purpose vehicle* (SPV), and cash flows from assets are allocated to *tranches*.
- Each tranche is defined in terms of upper (*detachment*) and lower (*attachment*) points representing the percentage of total notional.
- Cash flows are allocated to tranches by specifying what is known as a "waterfall": losses are applied in reverse order of seniority of tranches.


ABS

Asset 1
Asset 2
Asset 3
…
…
…
…
…
…
Asset *n*

Principal:
\$100 million

SPV

Senior tranche
Principal: \$75 million
Return = 6%

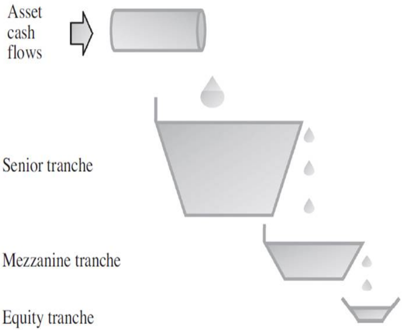
Mezzanine tranche
Principal: \$20 million
Return = 10%

Equity tranche
Principal: \$5 million
Return = 30%

§7 Pricing Collateralized Debt Obligation (CDO)

The Waterfall in ABS Cash flows

- *Equity tranche* is much less likely to realize its return than the other two tranches.
- There is a separate cash-flow waterfall for interest and principal:
    - *Interest cash flows* from the assets are allocated to *senior tranche* until *senior tranche* has received its promised return on its outstanding principal.
    - If promised return to the *senior tranche* can be made, cash flows are then allocated to *mezzanine tranche*.
    - *Principal cash flows* are used first to repay the principal of *senior tranche*, then *mezzanine tranche*, and finally *equity tranche*.



Credit Ratings of ABS Tranches

- *Senior tranche* of ABS is designed to be rated AAA.
- *Mezzanine tranche* is typically rated BBB.
- *Equity tranche* is typically unrated.
- Unlike the ratings assigned to bonds, the ratings assigned to tranches are "negotiated ratings".
- The creator of ABS makes a profit when the total return on underlying assets is greater than the total return offered to tranches.

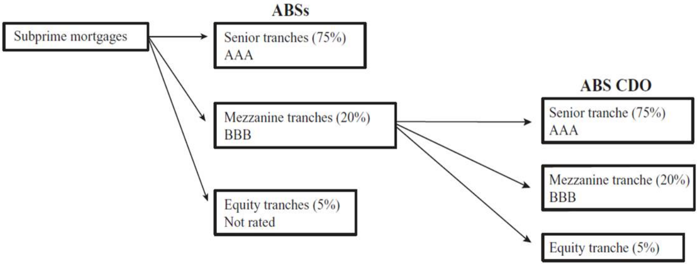
§7 Pricing Collateralized Debt Obligation (CDO)

ABS CDOs (Mezz CDOs)

- Senior AAA-rated tranches created from *subprime mortgages* can be easily sold to investors.
- *Equity tranches* are typically retained by the originator of mortgages or sold to a hedge fund.
- *Mezzanine tranches* are usually hard to sell.
- This led financial engineers to create an ABS from *mezzanine tranches* of ABSs that were original created from subprime mortgages.

§7 Pricing Collateralized Debt Obligation (CDO)

Losses to AAA Tranche of ABS CDO



Subprime mortgages

ABSs

Senior tranches (75%)
AAA

Mezzanine tranches (20%)
BBB

Equity tranches (5%)
Not rated

ABS CDO

Senior tranche (75%)
AAA

Mezzanine tranche (20%)
BBB

Equity tranche (5%)

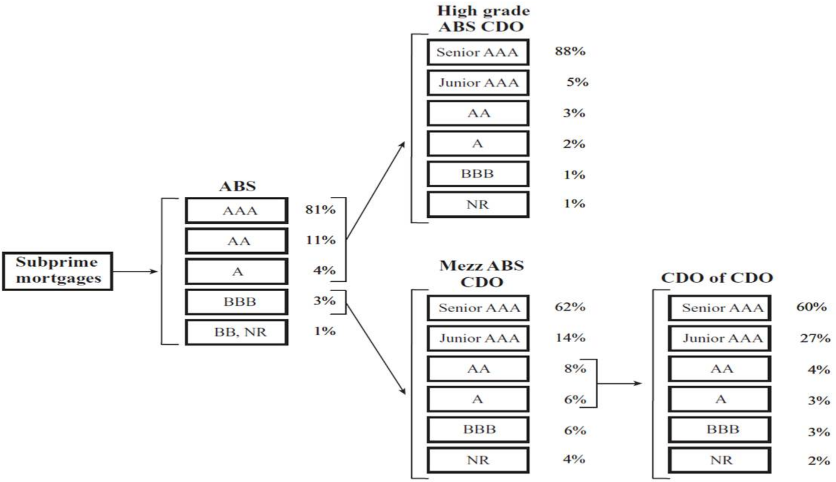
Losses to AAA-Rated Tranches of ABS CDO

Losses to Subprime Portfolios	Losses to Mezzanine Tranche of ABS	Losses to Equity Tranche of ABS CDO	Losses to Mezzanine Tranche of ABS CDO	Losses to Senior Tranche of ABS CDO
10%	25%	100%	100%	0%
15%	50%	100%	100%	33%
20%	75%	100%	100%	67%
25%	100%	100%	100%	100%

§7 Pricing Collateralized Debt Obligation (CDO)

Example of ABS CDOs

- More realistic example of subprime securitizations with ABS, ABS CDOs, and a CDO of CDO being created:



The Risk of BBB Tranches

- BBB tranches of ABSs are often quite thin (1%-3%).
- They tend to be either safe or completely wiped out.
- The rating agency models attempted to assign BBB tranche of ABS with the same probability of loss, i.e. the same expected loss, as a BBB bond.
- They have a quite different loss distribution (and correlation) from BBB bonds, and should not be treated as equivalent to BBB bonds (Coval et al., 2009a,b).

Dr. David Li, His Copula Models for Pricing CDOs, and Financial Innovations

- *"Understanding the credit risk profile of CDO tranches poses challenges even to the most sophisticated participants."* – Dr. Alan Greenspan, former chairman of US Federal Reserve (Financial Times, 2005)
- Dr. David Li invented the formula in his paper "*On Default Correlation: A Copula Function Approach*" (Li, 2000) for pricing CDOs which later "killed" Wall Street.
- Financial Times called him "the world's most influential actuary".

§7 Pricing Collateralized Debt Obligation (CDO)

Correlation Examples: Independent Defaults v.s. Perfectly-correlated Defaults

Consider a CDO with 100 bonds. Assume default rate on bonds is about 1% per year.

- **Independent defaults:**
    - Assume that defaults are independent (no clustering).
    - Each year, there will be about 1 default.
    - Over 5 years, there will be about 5 defaults.
    - This will almost certainly wipe out the entire equity tranche.
- **Perfectly-correlated defaults:**
    - Assume defaults are perfectly correlated: when one bond defaults, they all default.
    - Now, in about 1 year out of 100, everyone defaults; and in 99 years out of 100, no one defaults.
    - Over a 5-year period, there is about a 5% chance everyone defaults.
    - 5% of the time the equity tranche is wiped out, 95% of the time they suffer no loss.

§7 Pricing Collateralized Debt Obligation (CDO)

Standardised Synthetic CDOs: iTraxx EUR

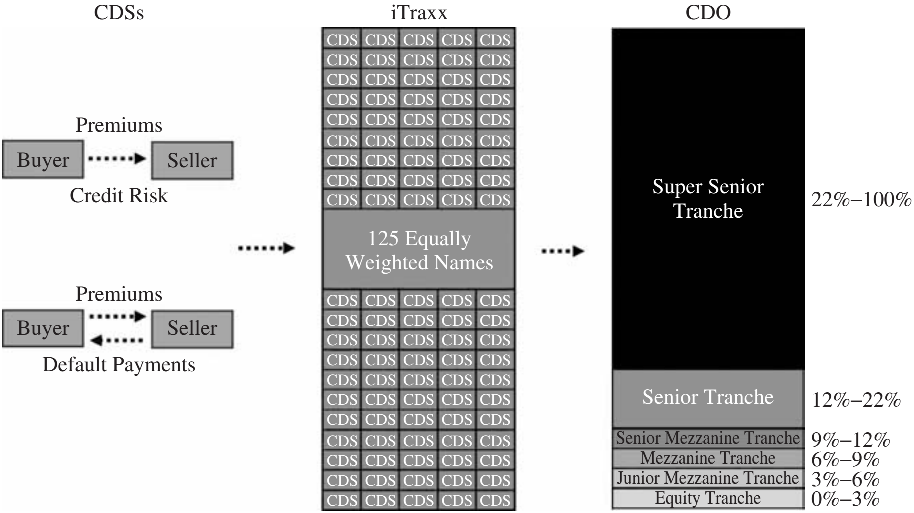


Figure: CDO mechanism and CDO tranches

§7 Pricing Collateralized Debt Obligation (CDO)

Market Quotes of Typical Standardised Synthetic CDO

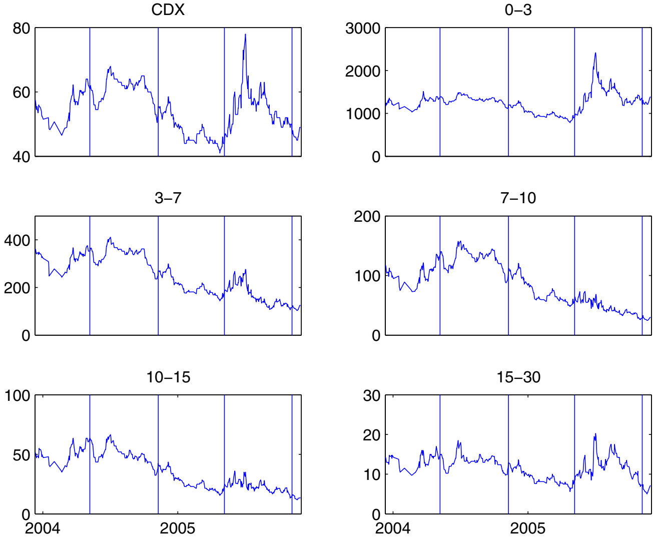


Figure: Time series of CDX index and tranche spreads (bps), 8/2003-10/2005 (Longstaff and Rajan, 2008)

Cash-flow Structure of A Synthetic CDO

- Consider a synthetic CDO with maturity T and underlying n different CDSs of the same maturity T, and same coupon-payment dates $0 < t_1 < t_2 < \cdots < t_m = T$.

- τ_i^* is denoted as default time of i^{th} name, $i = 1, 2, ..., n$.

- The accumulated (aggregated) portfolio loss process up to time t is

$$L_t = \sum_{i=1}^{n} N(1 - R_i)\mathbb{1}\{\tau_i^* \leq t\}, \qquad t \in [0, T],$$

  where N is notional and R_i is constant recovery rate of i^{th} name.

- The process of accumulated (aggregated) portfolio loss percentage up to time is

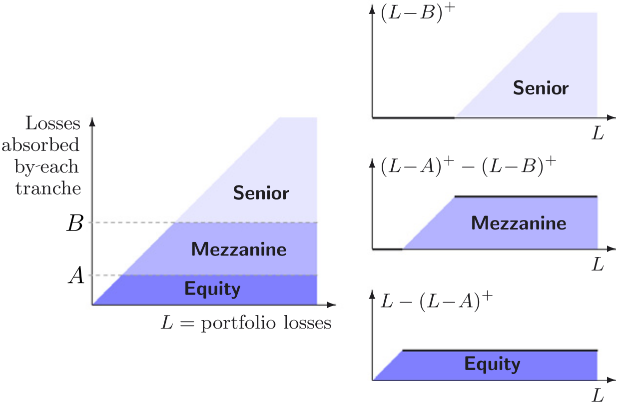
$$L_t^{\%} = \frac{L_t}{nN}.$$

§7 Pricing Collateralized Debt Obligation (CDO)

Cash-flow Structure of A Synthetic CDO

- The accumulated loss process of a CDO tranche of attachment A and detachment B up to time t is

$$L_t^{[A,B]} = (L_t - A)^+ - (L_t - B)^+, \qquad 0 \leq A < B, \quad t \in [0, T].$$



§7 Pricing Collateralized Debt Obligation (CDO)

Cash-flow Structure of A Synthetic CDO

For $A > 0$, from *protection sellers*' point of view (ignoring *counterparty risk*):

- **Expected loss of tranche $[A, B]$:**

$$\begin{aligned} Loss\ Leg_{[A,B]} &= \mathbb{E}\left[\int_0^T D(0,t)\,\mathrm{d}L_t^{[A,B]}\right] \\ &= \mathbb{E}\left[\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} D(0,t)\,\mathrm{d}L_t^{[A,B]}\right] \\ &\approx \mathbb{E}\left[\sum_{k=1}^{m} D\left(0, \frac{t_{k-1}+t_k}{2}\right)\left(L_{t_k}^{[A,B]} - L_{t_{k-1}}^{[A,B]}\right)\right], \end{aligned}$$

  where

  - $\mathrm{d}L_t^{[A,B]}$ is loss increment of tranche $[A, B]$ at time t;
  - $D(0, t)$ is the current price of a default-free zero-coupon bond of maturity t;
  - It is usually assumed that defaults only occur in the middle of coupon-payment dates (Andersen et al., 2003).

- **Expected premium of tranche $[A, B]$:**

$$\begin{aligned} \textit{Premium Leg}_{[A,B]} &= \mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k) \int_{t_{k-1}}^{t_k} s_0^{[A,B]} O_t^{[A,B]} \mathrm{d}t\right] \\ &\approx \mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k) s_0^{[A,B]} (T_k - T_{k-1}) \frac{O_{t_{k+1}}^{[A,B]} + O_{t_k}^{[A,B]}}{2}\right] \\ &= \mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k) s_0^{[A,B]} (T_k - T_{k-1}) \left(B - A - \frac{L_{t_k}^{[A,B]} + L_{t_{k-1}}^{[A,B]}}{2}\right)\right], \end{aligned}$$

where

- outstanding notional of tranche $[A, B]$ up to time t is

$$O_t^{[A,B]} = (B - A) - L_t^{[A,B]};$$

- $s_0^{[A,B]}$ is *credit spread* of tranche $[A, B]$ at today $t = 0$ when the contract is created.

§7 Pricing Collateralized Debt Obligation (CDO)

Cash-flow Structure of A Synthetic CDO

- Today $t = 0$, set fair spread $s_0^{[A,B]}$ such that the tranche' PV = *expected premium* – *expected loss* =0, then,

$$s_0^{[A,B]} \approx \frac{\mathbb{E}\left[\sum_{k=1}^{m} D\left(0, \frac{t_{k-1}+t_k}{2}\right)\left(L_{t_k}^{[A,B]} - L_{t_{k-1}}^{[A,B]}\right)\right]}{\mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k)(T_k - T_{k-1})\left(B - A - \frac{L_{t_k}^{[A,B]}+L_{t_{k-1}}^{[A,B]}}{2}\right)\right]}, \qquad A > 0;$$

or, simply,

$$s_0^{[A,B]} \approx \frac{\mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k)\left(L_{t_k}^{[A,B]} - L_{t_{k-1}}^{[A,B]}\right)\right]}{\mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k)(T_k - T_{k-1})\left(B - A - L_{t_k}^{[A,B]}\right)\right]}, \qquad A > 0.$$

§7 Pricing Collateralized Debt Obligation (CDO)

Cash-flow Structure of A Synthetic CDO

For $A = 0$, i.e. equity tranche $[0, B]$:

- Seller of equity tranche pays an up-front fee at the effective date of CDO and pays coupons at a fixed running spread of 500 bps per year to buyer.

- Equity tranche spread is defined as the ratio of up-front fee to the notional of equity tranche, i.e.

$$\begin{aligned} s_0^{[0,B]} \quad \approx \quad & \frac{1}{B}\left\{ \mathbb{E}\left[\sum_{k=1}^{m} D\left(0, \frac{t_{k-1} + t_k}{2}\right) \left(L_{t_k}^{[0,B]} - L_{t_{k-1}}^{[0,B]} \right) \right] \right. \\ & \left. -5\% \times \mathbb{E}\left[\sum_{k=1}^{m} D(0, t_k)(T_k - T_{k-1}) \left(B - \frac{L_{t_k}^{[0,B]} + L_{t_{k-1}}^{[0,B]}}{2} \right) \right] \right\}. \end{aligned}$$

§7 Pricing Collateralized Debt Obligation (CDO)

Typical Standardised Synthetic CDO Quotes

- Markets quote CDO tranches only for standardized pools of CDS.
- The most liquid indices:
  1. *iTraxx EUR* on 125 European names;
  2. *CDX IG* on 125 US names.

Table: Typical CDO Quotes for 5-year Tranches, Aug. 4, 2004 (Hull and White, 2004)

Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	0-100% (Index)
CDX IG	41.38% +500	349	135.5	46	14	63.25
iTraxx EUR	27.6% +500	168	70	43	20	42

Table: Implied Correlations from Gaussian-copula Model, Aug. 4, 2004

Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	Intensity λ
CDX IG	21.7%	4.1%	17.8%	18.5%	29.8%	1.066%
iTraxx EUR	20.5%	5.2%	16.1%	23.3%	31.2%	0.701%

Recommended Books

1. *Credit Risk: Pricing, Measurement, and Management* (Duffie and Singleton, 2003)
2. *Credit Risk Modeling: Theory and Applications* (Lando, 2004)
3. *Credit Derivatives Pricing Models: Models, Pricing and Implementation* (Schönbucher, 2003)
4. *Introduction to Credit Risk Modeling* (Bluhm et al., 2010)
5. *Credit Risk: Modeling, Valuation and Hedging* (Bielecki and Rutkowski, 2004)
6. *Credit Risk Modeling Using Excel and VBA* (Löeffler and Posch, 2010)
7. *Modelling Single-name and Multi-name Credit Derivatives* (O'Kane, 2011)

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