Teaching project of Abstract Algebra

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Contents

Chapter 1

Sets

1.1 Set Theory

I. Teaching objective:

The teaching objective of set theory is to provide students with an understanding of the fundamental concepts and principles that underline the study of sets and their properties. This includes a focus on set operations, relations between sets, the properties of sets, and applications of set theory in other areas of mathematics and computer science. Ultimately, the goal of this course is to equip students with the tools they need to analyze and solve problems using the concepts and techniques of set theory.

II. Teaching plan:

Strengthen students of the concept of set and objects. Review the operators of sets. To Understand the laws of sets. To master the concept and properties of power sets. In summary, understanding the concept and properties of power sets is essential in set theory and is used in various mathematical applications, such as in probability and combinatorics.

III. Teaching context:

1. A **set** is a well-defined collection of objects. The objects that belong to a set are called **elements**. We will denote sets by capital letters such as $A, B, C, \cdots$ We write $a \in A$ to mean that a is an element of the set A.

2. Laws of sets.

Let A, B and C be sets. Then

(1) Commutative law of union: $A \cup B = B \cup A$.

(2) Commutative law of intersection: $A \cap B = B \cap A$.

(3) Associative law of union: $(A \cup B) \cup C = A \cup (B \cup C)$.

(4) Associative law of intersection: $(A \cap B) \cap C = A \cap (B \cap C)$.

(5) Distributive law of intersection with respect to union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(6) Distributive law of union with respect to intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(7) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

(8) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(9) $(B \cup C) \backslash A = (B \backslash A) \cup (C \backslash A)$.

(10) $(B \cap C) \backslash A = (B \backslash A) \cap (C \backslash A)$.

3. The power set of a set X is defined as the set of all possible subsets of X, including the empty set and X itself. Let $X = \{1, 2\}$, then $P(X) = \{\emptyset, X, \{1\}, \{2\}\}$.

IV. Teaching time:

2 classes.

1.2 Maps

I. Teaching objective:

The teaching objective of maps is to provide students with an understanding of maps. This includes maps, operations of maps and injective map, surjective map and bijective map.

II. Teaching plan:

To provide some concrete examples to help students better understand the concept and properties of power sets. Understanding the concepts and properties of power sets is important and has many applications.

III. Teaching content:

1. A map f from a nonempty set X to a nonempty set Y is a law which assigns to each $x \in X$ exactly one element $y = f(x) \in Y$. Written $f : X \longrightarrow Y : x \longmapsto y$. The set X is the ***domain*** of f, Y is the ***codomain*** of f, $f(x)$ is called the ***image*** of x, x is called the ***preimage*** of $f(x)$. Denote the set of images as $\mathrm{Im}(f)$.

2. Some particular maps: identity map, constant map, conclusion map, restriction map.

3. Main definitions and conclusions.

(1) A map $f : X \longrightarrow Y$ is said to be ***injective*** if given any $x, x' \in X$, $x \neq x'$ implies that $f(x) \neq f(x')$.

(2) A map $f : X \longrightarrow Y$ is said to be ***surjective*** if $\mathrm{Im}(f) = Y$.

(3) A map $f : X \longrightarrow Y$ is called ***bijective*** if f is both injective and surjective.

(4) Let $f : X \longrightarrow Y$ be a map. The map $g : Y \longrightarrow X$ is called a ***left inverse*** of f if $g \circ f = 1_X$. The map $h : Y \longrightarrow X$ is called a ***right inverse*** of f if $f \circ h = 1_Y$.

(5) A map $g : Y \longrightarrow X$ is called a ***two-sided inverse*** if g is a left inverse and a right inverse of f, i.e.

$$g \circ f = 1_X, \; f \circ g = 1_Y.$$

(6) Let $f : X \longrightarrow Y$ be a map , then f is injective if and only if f has (at least) a left inverse; f is surjective if and only if f has (at least) a right inverse.

(7) Let $f : X \longrightarrow Y$ be an injective map and $(A_i)_{i \in I}$ be a family of subsets of X. Then

$$f\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in I} f(A_i).$$

IV. Teaching time:

2 classes.

1.3 Equivalence relations and equivalence classes

I. Teaching objective:

Equivalence classes and equivalence relations are important topics in mathematics and computer science. An equivalence relation is a binary relation that is reflexive, symmetric, and transitive, meaning it satisfies the following properties:

An equivalence relation partitions a set into equivalence classes or subsets of elements that are related to each other by the relation. Each equivalence class contains all the elements that are related to one another, and every element belongs to one and only one equivalence class.

Understanding equivalence classes and equivalence relations is essential for many mathematical and computational concepts, including quotient groups, modular arithmetic, and partitioning data structures.

II. Teaching plan:

Equivalence class and equivalence relation are important concepts in mathematics and computer science. The teaching philosophy behind these concepts emphasizes the importance of grouping elements that share some common attribute or relationship. By focusing on the relationships between elements, we can simplify complex problems and identify patterns that help us understand the structure of the underlying system. By treating elements within these sets as equivalent according to certain criteria, we can design algorithms that are efficient and effective. This approach also allows us to identify errors or inconsistencies in the data, which can improve the accuracy of our computational models.

Overall, the teaching philosophy behind equivalence classes and equivalence relations emphasizes the importance of abstraction and pattern recognition in problem-solving. By focusing on the relationships between elements rather than their individual characteristics, we can gain valuable insights into the underlying structures of complex systems.

III. Teaching content:

1. Let X and Y be sets. We can define a new set

$$X \times Y = \{(x, y) | x \in X, y \in Y\}.$$

$X \times Y$ is called the ***Cartesian product*** of X and Y. And (x, y) is called an

ordered pair, where x is the first component, y is the second component.

Let X and Y be sets. A ***relation*** R between X and Y is a subset of $X \times Y$.

2. Let X be a set, $x, y, z \in X$. R is a relation of $X \times X$.

(1) R is called ***reflexive*** if $(x, x) \in R$;

(2) R is called ***symmetric*** if $(x, y) \in R$ implies $(y, x) \in R$;

(3) R is called ***transitive*** if $(x, y), (y, z) \in R$ imply $(x, z) \in R$.

If R is reflexive, symmetric and transitive, it is called an ***equivalence relation***.

3. A ***partition*** $\mathfrak{P}$ of a set X is a collection of sets $X_1, X_2, \cdots, X_n$, such that $X_i \bigcap X_j = \emptyset, i \neq j$ and $\bigcup_i X_i = X$.

4. Let $\sim$ be an equivalence relation on a set X. Then the equivalence classes of X from a partition of X. Conversely, if $\mathfrak{P} = \{X_i\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_i.

(5) Let r and s be two integers and suppose that $n \in \mathbb{N}$. We say that r is **congruent** to s modulo n if $r - s$ is divisible by n, $r - s = nk$ for some $k \in \mathbb{Z}$. We write $r \equiv s(\mod n)$. For example,

$$5 \equiv 11(\mod 3),$$

because $5 - 2 = 3$, and $-1 - 2 = 3 \times (-1)$. Thus 5 and -1 have the same remainder, that is $5(\mod 3) = 11(\mod 3)$. Congruence modulo n forms an equivalence relation of $\mathbb{Z}$.

IV. Teaching time:

4 classes.

1.4 Factor sets

I. Teaching objective:

The teaching goals for quotient sets emphasize the importance of abstraction and pattern recognition in problem-solving, and aim to equip students with the skills and knowledge necessary to apply these concepts in a variety of settings. Emphasize the importance of understanding how to group elements of a set based on a given equivalence relation. Simplify complex problems and identify patterns that help us understand the structure of the underlying system.

II. Teaching plan:

To help students understand how to define a partition of a set based on a given equivalence relation, and how to use that partition to create quotient sets. This involves learning how to identify the equivalence classes within a given equivalence relation and understanding how they relate to the entire set.

Another important goal is to help students understand how to use quotient sets to solve problems in a wide range of mathematical and computational contexts. This may include applications in algebra, topology, geometry, and computer science, among others. By mastering the concepts of quotient sets, students can gain valuable skills that will be useful in a wide range of fields.

III. Teaching content

1. A ***factor set*** of X about an equivalence relation R is a set which elements are equivalence classes, denoted by X/R.

2. Define

$$a(\mod n) + b(\mod n) = a + b(\mod n), \tag{1.1}$$

$$(a(\mod n)) \cdot (b(\mod n)) = ab(\mod n). \tag{1.2}$$

3. Let $\overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_n$. Then

$$\begin{aligned}
&\overline{a} + \overline{b} = \overline{b} + \overline{a},\\
&\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a},\\
&(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c}),\\
&(\overline{a} \cdot \overline{b}) \cdot \overline{c} = \overline{a} \cdot (\overline{b} \cdot \overline{c}),\\
&\overline{a} + \overline{0} = \overline{a},\\
&\overline{a} \cdot \overline{1} = \overline{a},\\
&\overline{a} \cdot (\overline{b} + \overline{c}) = \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c},\\
&\overline{a} + (\overline{-a}) = \overline{0}.
\end{aligned}$$

4. A ordered pair (X, R) is called a ***partial-ordered set*** if X is a set and R is a partial relation on X. Most of time, a partial ordering set is denoted by $(X, \preceq)$. If for all $x, y \in X$, either xRy or yRx, that is whenever any two elements are comparable, (X, R) is called a ***totally-ordered set***.

5. A ***well-ordered*** on a set X is a partial ordering such that every nonempty subset A of X has a smallest element a. That is $a \leq b$ for every $b \in A$.

6. $\mathbb{N}$, $\mathbb{Z}^+$ are well-ordered sets. But $\mathbb{Z}$ is not a well-ordered set.

7. Every nonempty subset of the natural numbers is well-ordered.

IV. Teaching time:

1 class.

1.5 Arithmetic

I. Teaching objective:

The teaching of division algorithm is aimed at providing students with a valuable problem solving tool to tackle sophisticated algebraic problems. By mastering this technique, students are well equipped to approach more complex topics, including factoring, solving equations, and graphing polynomial functions.

The Fundamental Theorem of Algebra is a critical concept in Algebra that students need to understand to excel in advanced mathematics. The teaching of this theorem is focused on instilling in students the understanding that any polynomial of degree n has n complex roots.

II. Teaching plan:

Explain that the Fundamental Theorem of Algebra implies that every polynomial of degree n has n roots in the set of complex numbers. Demonstrate some practical applications, such as finding the roots of a quadratic equation using the quadratic formula. This would help students see the practical importance of learning the Fundamental Theorem of Algebra.

Encourage students to ask questions and engage in discussions during the lesson to deepen their understanding of the theorem. Encourage students to practice their math skills continuously and apply the theorem in exercise problems. Students can develop the mathematical reasoning and problem-solving skills required to excel in advanced mathematics.

III. Main teaching content

1. **Division Algorithm:** Let a and b be integers, with $b > 0$. Then there exists an unique integer q and r such that $a = bq + r$, where $0 \leqslant r < b$.

2. The **greatest common divisor** of integers a and b is a positive integer d such that d is a common divisor of a and b, and if d' is any other common divisor of a and d, then $d'|d$. We write $d = gcd(a, b)$.

3. Let a and b be nonzero integers. Then there exist integers r and s such that $gcd(a, b) = ar + bs$. Furthermore, the greatest common divisor of a and b is unique.

4. There exist infinite numbers of primes.

5. **Fundamental theorem of Arithmetic:** Let n be an integer and $n > 1$. Then

$$n = p_1 p_2 \cdots p_k,$$

where $p_1, p_2, \cdots, p_k$ are primes(not necessary distinct). Furthermore, this factorization is unique, that is if $n = q_1 q_2 \cdots q_l$, then $k = l$, and the q_i's are just the p_i's rearranged.

IV. Teaching time:

1 class.

1.6 The Chinese Remainder Theorem

I. Teaching objective:

The Chinese Remainder Theorem is a critical concept in Number Theory. The teaching of this theorem is focused on instilling in students the understanding of how to solve the system of congruences, which is beneficial in cryptography, computer science, and other fields. The Chinese Remainder Theorem also has important implications in cryptography, where it is used to create more secure algorithms. By understanding the theorem, students can better appreciate the role of mathematics in cryptography and computer science.

II. Teaching plan:

The primary objective of teaching the Chinese Remainder Theorem is to educate students on how to simplify complex systems of congruences by breaking them down into simpler systems that can be more easily solved. Through the use of the theorem, students can develop essential computational and mathematical reasoning skills that are applicable in many different areas. Additionally, the teacher should encourage students to explore and apply the theorem in various exercises and real-world problems. The theorem's versatility enables students to solve problems that could not have been solved otherwise.

The objective of teaching the Chinese Remainder Theorem is to provide students with the skills and knowledge required to solve complex systems of congruences and apply this knowledge to areas such as cryptography and computer science. Through the use of examples, exercises, and real-world problems, students can develop their mathematical reasoning skills and prepare to excel in advanced mathematics.

III. Teaching content

1. Let a, b, n be integers with $n > 0$, $\gcd(a, n)$ be the greatest common divisor of a and n. Then there is a solution x of the congruence equation $ax = b(\mod n)$ if and only if $\gcd(a, n)|b$.

2. **The Chinese Remainder Theorem:** Let $a_1, a_2, \cdots, a_k$ and $n_1, n_2, \cdots, n_k$ be integers with $n_i > 0$; assume that $gcd(n_i, n_j) = 1$ if $i \neq j$. Then there is a common solution x of the system of congruence equations:

$$\begin{cases} x = a_1(\mod n_1) \\ \vdots \\ x = a_k(\mod n_k) \end{cases}$$

3. It is well know that an integer is divisible by 3 if and only if the sum of its digits is a multiple of 3. Let $m = m_k m_{k-1} \cdots m_1 m_0$ be the decimal representation of an integer m, where $0 \leq m_i \leq 9$. Then

$$m = m_k 10^k + m_{k-1} 10^{k-1} + \cdots m_1 10 + m_0.$$

Note that $10 = 1(\mod 3)$, $\bar{10} = \bar{1}$. Then $\bar{10}^i = \bar{1}^i = \bar{1}$ for $i \geq 0$. It follows that $m = m_k + m_{k-1} + \cdots + m_1 + m_0(\mod 3)$. This finished the proof.

IV. Teaching time:

2 classes.

Chapter 2

Group

2.1 Definitions and examples

I. Teaching objective:

Group Theory is an essential component of higher mathematics and has numerous applications to many fields of research. The fundamental objective of teaching Group Theory is to equip students with a deep understanding of abstract algebraic structures. The primary objective of teaching Group Theory is to enable students to develop and apply mathematical reasoning and problem-solving skills. Group Theory provides a gate into abstract algebra, enhancing students' analytical thinking capacity and providing a framework for many mathematical concepts. Group Theory enhances students' understanding of symmetry and pattern recognition, which are ubiquitous in nature and scientific discoveries. With a solid understanding of Group Theory, students can comprehend and predict the behavior of complex systems, leading to better scientific research outcomes.

II. Teaching plan:

Teaching Group Theory is also vital for students pursuing careers in mathematical fields such as pure mathematics, physics, and computer science. Group Theory provides the foundation upon which various mathematical concepts can be built, including Lie Theory, Cohomology Theory, Topology, and others. The subject introduces students to the fundamental concepts of abstract algebra, encourages problem-solving skills, enhances their understanding of symmetry, and provides the foundation for advanced mathematical fields. Therefore, the incorporation of Group Theory in mathematics curricula is of utmost importance, and its teaching should be prioritized.

III. Teaching content:

1. A binary **operation** of a set X is a function

$$X \times X \longrightarrow X,$$

for any $(x, y) \in X \times X$ a unique element $x \circ y$ or xy in X.

2. A ***semigroup*** $(G, \circ)$ is a set G with a binary operation $G \times G \longrightarrow G$, that satisfies the associative law:

$$(a \circ b) \circ c = a \circ (b \circ c), \text{ for } a, b, c \in G$$

3. A ***group*** $(G, \circ)$ is a set G with a binary operation $G \times G \longrightarrow G$, that satisfies the following axioms:

(1) The binary operation is associative. that is $(a \circ b) \circ c = a \circ (b \circ c)$ for $a, b, c \in G$.

(2) There exists an element $e \in G$, called the ***identity element***, such that for any element $a \in G$, $a \circ e = a = e \circ a$.

(3) For each element $a \in G$, there exists an ***inverse element*** in G, denoted by a^{-1}, such that $a \circ a^{-1} = e = a^{-1} \circ a$.

A group G with the property that $a \circ b = b \circ a$ for all $a, b \in G$ is called **abelian group** or **commutative group**. Groups not satisfying this property are said to be **nonabelian group** or **noncommutative group**.

4. Let $M_2(\mathbb{R})$ be the set of 2×2 matrices over $\mathbb{R}$, $GL_2(\mathbb{R})$ be the subset of $M_2(\mathbb{R})$ consisting of invertible matrices. Then $(GL_2(\mathbb{R}), \cdot)$ is a group with matrices product, called the ***general linear group***.

5. The **order** of a finite group is the number of elements that it contains. If G is a group containing n elements, we write $|G| = n$. The group $\mathbb{Z}_3$ is a finite group of order 3.

6. Let G be a group and a and b be any two elements in G, then the equations $ax = b$ and $ya = b$ have unique solution in G.

IV. Teaching time:

2 classes.

2.2 Subgroups

I. Teaching objective:

Subgroups is a vital component of group theory, which is a critical branch of abstract algebra. The main goal of teaching subgroups is to equip students with a solid understanding of the concepts and properties of subgroups.

Through the study of subgroups, students will learn how to identify subgroups of a given group, understand the relationship between subgroups and quotient groups, and explore the isomorphism theorems of groups. These skills are essential for further studies in abstract algebra and related fields, including algebraic topology, number theory, and algebraic geometry.

II. Teaching plan:

By introducing the concept of a subgroup and the fundamental properties of subgroups, including the identity element, inverse elements, and closure under the group operation. Students should be provided with numerous examples of subgroups and encouraged to identify subgroups in familiar groups, such as permutation groups and matrix groups.

Next, students should be introduced to the Lagrange's theorem, which states that the order of a subgroup divides the order of the group. This theorem provides a powerful tool for identifying subgroups of finite groups and plays a critical role in many areas of mathematics.

Finally, students should learn about the isomorphism theorems of groups, which enable them to understand the relationship between subgroups and quotient groups. This knowledge helps students to grasp the deep connections between different groups, identify isomorphic groups, and gain insights into the structure and properties of groups.

Overall, teaching subgroups is a crucial aspect of group theory, and a well-designed teaching plan should equip students with the necessary skills and knowledge to identify, understand, and analyze subgroups and their properties.

III. Teaching content:

1. A subgroup H of a group $(G, \circ)$ to be a subset H of G such that H is a group under the group operation of G.

2. Let $SL_n(\mathbb{R}) = \{A|A \in GL_n(\mathbb{R}), |A| = 1\}$ be the subset of $GL_n(\mathbb{R})$. Note that the product of two matrices of determinant one also has determinant one. The group $SL_2(\mathbb{R})$ is called the ***special linear group***. $SL_n(\mathbb{R})$ is a proper subgroup of $GL_n(\mathbb{R})$.

3. A subset H of G is a subgroup if and only if it satisfies the following conditions.

(1). The identity e of G is in H.

(2). If $h_1, h_2 \in H$, then $h_1h_2 \in H$.

(3). If $h \in H$, then $h^{-1} \in H$.

4. Let H be a subset of a group G. Then H is a subgroup of G if and only if H is nonempty, and whenever $g, h \in H$, then $gh^{-1} \in H$.

IV. Teaching time:

2 classes.

2.3 Cyclic groups

I. Teaching objective:

Cyclic groups are an essential concept in abstract algebra, and the study is fundamental to understanding group theory. The primary goal of teaching cyclic groups is to provide students with a solid understanding of the fundamental properties of cyclic groups and to equip them with the necessary skills to analyze and classify cyclic groups.

The teaching of cyclic groups involves a variety of key concepts, including generators, orders, subgroups, and homomorphisms. By the end of the course, students should understand the classification of finite cyclic groups and the uniqueness of the infinite cyclic group up to isomorphism.

II. Teaching plan:

Introduction to the concept of cyclic groups, including their definition and basic properties. This should involve an explanation of generators and the cyclic subgroup generated by an element. Next, students should learn about the order of a cyclic group and how to determine the orders of its elements. This will involve introducing the concept of the Euler totient function, which is used to find the number of elements of a given order in a cyclic group. After this, students should be introduced to the structure of finite cyclic groups, including their classification and examples. This will involve exploring the factors of the order of cyclic groups and understanding the structure of the multiplicative group of a finite field. Finally, the teaching plan should conclude with a discussion of infinite cyclic groups, including their classification and examples.

Students should also be given ample opportunities to practice their skills through a variety of examples, exercises, and problems. This will allow them to demonstrate their mastery of the concepts and properties of cyclic groups and apply them to real-world problems.

III. Teaching content:

1. Let G be a group, $g \in G$. Then the set $\langle g \rangle = \{g^k | k \in \mathbb{Z}\}$ is a subgroup of G. Furthermore, $\langle g \rangle$ is the smallest subgroup of G contains g.

2. The set $\langle g \rangle$ is called the ***cyclic subgroup*** generated by g. If $G = \langle g \rangle$, then G is called a cyclic group, g is a ***generator*** of G.

3. Groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are cyclic groups. And $\mathbb{Z} = \langle 1 \rangle$, $\mathbb{Z}_n = \langle \overline{1} \rangle$. Moreover, $\mathbb{Z} = \langle -1 \rangle$, $\mathbb{Z}_n = \langle \overline{-1} \rangle$. Notice that a cyclic group can have more than a single generator.

4. Every cyclic group is abelian.

5. Every subgroup of a cyclic group is cyclic.

6. Let $G = \langle a \rangle$ be a cyclic group.

(1) If G is infinite, then each subgroup of G has the form $G_m = \langle a^m \rangle$ where $m \geq 0$. Furthermore, the G_m are all distinct and G_m has infinite order if $m > 0$.

(2) If G has finite order n, then it has exactly one subgroup of order d for each positive divisor d of n, namely $< a^{\frac{n}{d}} >$.

IV. Teaching time:

2 classes.

2.4 Permutation groups

I. Teaching objective:

Permutation groups are a fundamental concept in abstract algebra, and it is essential to understanding group theory. The primary goal of teaching permutation groups is to provide students with a solid understanding of the definition and basic properties of permutation groups and to equip them with the necessary skills to analyze and classify these groups. The study of permutation groups involves a variety of key concepts, including cycles, transpositions, and permutation sign. By the end of the course, students should be able to classify permutations groups and understand its relationship with other mathematical concepts such as group homomorphisms and group actions.

II. Teaching plan:

Begin with an introduction to the concept of permutation groups, including definitions, basic terminology, and properties. This should involve an explanation of the structure of permutations and the key properties of this type of group. Students should learn about the cycle notation, cycle decomposition, and the order of a permutation. This will involve exploring the different ways of representing permutations and decomposing them into cycles. The teaching plan should conclude with a discussion of transpositions and symmetry, including a proof of the sign of a permutation group. Student will to demonstrate their mastery of the concepts and properties of permutation groups and apply them to real world problems.

III. Teaching content:

1. A **permutation** of a set X is a bijection from X to X. Assume that $X = \{x_1, x_2, \cdots, x_n\}$, σ is a permutation of X, then $\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_n)$ are all different and therefore constitute all n elements of the set X, but in some different order from $x_1, \cdots, x_n$. Let $|X| = n$, denote the set of all permutations of X as S_n. Next, we will show that S_n is a group.

2. The set S_n is a group with $n!$ elements. The operation of S_n is the composition of bijections.

3. Every permutation in S_n is expressible as a product of disjoint cycles and the cycles appearing in the product are unique.

4. Every nonidentity element of S_n is expressible as a product of transposi-

tions.

5. A permutation is called the ***even permutation*** if the permutation can be written as even number of transpositions. Similarly, the ***odd permutation*** is the permutation which can be written as odd number of transpositions.

6. The ***alternating group*** A_n is the set of all even permutations of S_n. The alternating group A_n is a subgroup of S_n. Moreover, the order of A_n is $\frac{n!}{2}$ for $n \geq 2$.

IV. Teaching time:

2 classes.

2.5 Dihedral groups

I. Teaching objective:

Dihedral groups are an essential concept in the study of abstract algebra, and their understanding is crucial to comprehend group theory. The primary objective of teaching dihedral groups is to provide students with an in-depth understanding of the definition and fundamental properties of dihedral groups and enable them to classify and analyze these groups. The study of dihedral groups involves several key concepts, including reflections, rotations, and symmetries. By the end of the course, students should be able to classify dihedral groups based on the number of sides of a polygon and understand the essential relationships between these groups and other mathematical concepts, such as group actions and homomorphisms.

II. Teaching plan:

Teaching for dihedral groups should start with an introduction to the concept of dihedral groups, including definitions and basic terminology. This should involve an explanation of the structure of dihedral groups, their important properties, and examples of how they can be used. Students should learn about the concept of symmetry through rotations and reflections, including an analysis of the properties of these operations, how they relate to dihedral groups, and how their compositions generate the dihedral group. Throughout the course, students should be given ample opportunities to practice their skills and apply their knowledge through a variety of examples, exercises, and problems. This will allow them to demonstrate their mastery of the concepts and properties of dihedral groups and their ability to apply them to real-world problems.

III. Teaching content:

1. The n-th ***dihedral group*** D_n is a group of rigid motions of a regular n-gon.

2. The dihedral group D_n, is a subgroup of S_n of order $2n$. And

$$D_n = < s, r | s^2 = 1, r^n = 1, srs = r^{-1} > .$$

3.$D_3 = < r, s \mid s^2 = (1) = r^3,\ srs = r^{-1} > = \{1, s, r, r^2, rs, r^2s\}$, $D_4 = < r, s \mid s^2 = id = r^4,\ srs = r^{-1} > = \{1, s, r, r^2, r^3, rs, r^2s, r^3s\}$.

IV. Teaching time:

1 classes.

Chapter 3

Properties of Groups

3.1 Cosets and Lagrange's Theorem

I. Teaching objective:

Lagrange's theorem is an essential concept in the study of abstract algebra, and its understanding is crucial to comprehend group theory. The primary objective of teaching Lagrange's theorem is to provide students with an understanding of the fundamental properties of subgroups and enable them to use Lagrange's theorem to calculate the orders of subgroups.

The study of Lagrange's theorem involves several key concepts, including the theorem's statement, proof, and applications. Students should be able to state and understand the significance of Lagrange's theorem, apply the theorem to calculate subgroup orders, and prove simple cases of the theorem.

II. Teaching plan:

Introduced to Lagrange's theorem, including its statement and proof. This should be followed by a discussion of the corollaries of the theorem, including the fact that the order of a subgroup must divide the order of the parent group, and a further discussion of the theorem's applications. Students should have an applying Lagrange's theorem to calculate subgroup orders and prove simple cases of the theorem. This can be done through a variety of examples, exercises, and problems.

Discussion of the significance of Lagrange's theorem in the larger context of group theory, including its role in the classification of groups and its relationship to other important concepts, such as the isomorphism theorems. Students demonstrate their mastery of the concepts and properties of Lagrange's theorem and their ability to apply them to real-world problems.

III. Teaching content:

1. Let G be a group, and H be a subgroup of G. Define a ***left coset*** of H with representative element $g \in G$ to be the set of

$$gH = \{gh|h \in H\},$$

define a ***right coset*** to be

$$Hg = \{hg|h \in H\}.$$

gH and Hg are subsets of G. If the left coset is equal to the right coset, it is called ***coset.***

2. Let H be a subgroup of a group G, then the group G is the disjoint union of the left coset of H in G.

3. ***Lagrange's Theorem*** Let G be a finite group and H be a subgroup of G. Then

$$\frac{|G|}{|H|} = [G : H].$$

In particular, the order H must divide the order of G.

4. The converse of Lagrange's Theorem is false. We will give a counter example. The group A_4 has no subgroup of order 6.

5. Some important results in number theory.

(1) The ***Euler function*** is the map $\Phi : \mathbb{N} \to \mathbb{N}$ defined by $\Phi(1) = 1$ and $\Phi(n)$ is the number of positive integers m with $1 \leq m < n$ and $gcd(m, n) = 1$. For example, $\Phi(3) = 2$, $\Phi(8) = 4, \Phi(12) = 4$.

(2) Let $U(n)$ be the group of units in $\mathbb{Z}_n$. Then $|U(n)| = \Phi(n)$.

(3) ***Euler's Theorem*** Let a and n be integers such that $n > 0$ and $gcd(a, n) = 1$. Then $a^{\Phi(n)} \equiv 1(\mod n)$.

(4) ***Fermat's little Theorem*** Let p be any prime numbers and suppose that $p \nmid a$, then $a^{p-1} \equiv 1(\mod p)$. Furthermore, for any integer b, $b^p \equiv b(\mod p)$.

IV. Teaching time:

2 classes.

3.2 Normal subgroups and factor groups

I. Teaching objective:

Normal subgroups is an essential component of group theory, and its understanding is crucial for students to comprehend several important concepts, such as group homomorphisms, quotient groups, and the isomorphism theorems. The primary objective of teaching normal subgroups is to provide students with a solid understanding of the definition, properties, and applications of normal subgroups in group theory.

The teaching on normal subgroups should start with an introduction to the concept of normal subgroups, including their definition, and the basic properties, such as the normality test. Focus on the essential applications of normal subgroups in group theory, such as group homomorphisms, quotient groups, and the isomorphism theorems.

II. Teaching plan:

Teaching normal subgroups should provide students with a comprehensive understanding of the definition, properties, and applications of normal subgroups in group theory. Students should have ample opportunities for both theoretical and practical learning, which will enable them to grasp complex concepts and apply them to problem-solving effectively.

III. Teaching content:

1. A subgroup H of G is called a ***normal subgroup*** in G if $gH = Hg$ for all $g \in G$.

2. If N is a normal subgroup of a group G, then the cosets of N in G form a group G/N under the operation $(aN)(bN) = abN$. This group is called the ***factor group*** or ***quotient group*** of G on N. Note that N is the identity of G/N. For any $bN \in G/N$, we have $(eN)(bN) = ebN = bN$.

3. Let N be a normal subgroup of G. The coset of N in G form a group G/N of order $[G : N]$.

4. Since $(\mathbb{Z}, +, 0)$ is an abelian group, all its subgroups are normal.

IV. Teaching time:

3 classes.

3.3 Homomorphisms of groups

I. Teaching objective:

Group homomorphisms is an essential component of group theory, and its understanding is crucial for students to grasp several critical concepts, such as the kernel, image, and the First Isomorphism Theorem. The primary objective of teaching group homomorphisms is to equip students with a solid understanding of the definition, properties, and applications of group homomorphisms in group theory.

Introducing the concept of mappings between groups, including the definition of homomorphisms, and their basic properties, such as the preservation of group operations. Then, students should be introduced to important examples of group homomorphisms, such as the determinant function.

II. Teaching plan:

The teaching should also include an extensive practice of solving problems related to group homomorphisms. These exercises can range from straightforward applications of the homomorphism definition to more complex group theory problems that require the use of theorems such as the First Isomorphism Theorem.

The teaching plan should conclude by highlighting the significance of group homomorphisms in the broader context of group theory. This should involve a discussion of how group homomorphisms relate to other essential concepts such as group isomorphisms, group automorphisms, and the classification of finite simple groups. Provide students with a thorough understanding of the definition, properties, and applications of group homomorphisms in group theory. Students should have ample opportunities for both theoretical and practical learning, which will enable them to grasp complex concepts and apply them to problem-solving effectively.

III. Teaching content:

1. A ***homomorphism*** between group $(G, \cdot)$ and $(H, \circ)$ is a map $\phi : G \longrightarrow H$, such that $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$. Images of ϕ in H is called the homomorphic image of ϕ.

2. Let $\phi : G \longrightarrow H$ be a group homomorphism and suppose that e is the identity of H. The ***kernel*** of ϕ is the subgroup $\{\phi^{-1}(e)\}$. Denote

$$\text{Ker}\phi = \{x | \phi(x) = e_H, x \in G\}.$$

3. Let $\phi : G \longrightarrow H$ be a homomorphism. Then the kernel of ϕ is a normal subgroup of G.

4. Let H be a normal subgroup of G. Define the ***natural*** or ***canonical homomorphism***. $\psi : G \longrightarrow G/H : g \longmapsto gH$.

5. Let $N = \{(1), (123), (132)\}$ be the normal subgroup of S_3, $\psi : S_3 \longrightarrow S_3/N$ be the natural homomorphism. Then $\text{Ker}\psi = N$.

IV. Teaching time:

2 classes.

3.4 Isomorphisms of groups

I. Teaching objective:

Group isomorphisms is a fundamental part of abstract algebra, and its understanding is essential for students to comprehend several critical concepts, such as the structure of finite groups, the classification of finite groups, and the concept of group actions. The primary objective of teaching group isomorphisms is to equip students with a solid understanding of the definition, properties, and applications of group isomorphisms in group theory.

Group isomorphisms should begin by introducing the definition of group isomorphism, and its basic properties such as bijectivity, preservation of group operations and inverses. Then, students should be introduced to the concept of isomorphic groups and the significance of isomorphisms in determining the structure of groups.

II. Teaching plan:

The teaching plan include extensive practice of problem-solving related to group isomorphisms. These exercises can range from straightforward applications of the isomorphism definition to more complex abstract algebra problems. Students understand the definition, properties, and applications of group isomorphisms in group theory. The students should have ample opportunities for both theoretical and practical learning, which will enable them to comprehend complex concepts and apply them effectively to problem-solving.

III. Teaching content:

1. Let $(G, \circ)$ and $(H, *)$ are two groups. If there exists a bijection $\phi : G \longrightarrow H$ such that

$$\phi(g_1 \circ g_2) = \phi(g_1) * \phi(g_2),\ \ g_1, g_2 \in G,$$

then ϕ is called a ***group isomorphism*** from G to H. We call G is isomorphic to H, denoted by $G \cong H$.

2. Let $\phi : G \longrightarrow H$ be an isomorphism of groups $(G, \cdot)$ and $(H, *)$. Then the following statements are true

(1) $\phi^{-1} : H \longrightarrow G$ is an isomorphism.

(2) $|G| = |H|$.

(3) If G is abelian, then H is abelian.

(4) If G is cyclic, then H is cyclic.

(5) If G has a subgroup of order n, then H has a subgroup of order n.

3. A cyclic groups of infinite order is isomorphic to $\mathbb{Z}$.

4. A cyclic groups of order n is isomorphic to $\mathbb{Z}_n$.

5. **Cayley Theorem** Every group is isomorphic to a group of permutations.

IV. Teaching time:

2 classes.

3.5 Fundamental Isomorphism theorems of group

I. Teaching objective:

Isomorphism theorems are fundamental part of abstract algebra and a critical component of students' foundational knowledge of algebraic structures. The primary objective of teaching the isomorphism theorems is to equip students with a solid understanding of the fundamental concepts, definitions, and applications of the isomorphism theorems in algebraic structures.

Students should learn the definition of the isomorphism theorem and the intuitive meaning behind it. This would involve discussing in detail the three parts of the isomorphism theorem: the first isomorphism theorem, the second isomorphism theorem, and the third isomorphism theorem in different algebraic structures.

II. Teaching plan:

Discuss of the applications and significance of the isomorphism theorems in abstract algebra and the wider field of mathematics. A variety of problem-solving exercises that range from applying the definitions of isomorphism theorems to complex abstract algebraic problems that require the use of advanced concepts such as Isomorphism Theorem. The emphasis should be on developing students' problem-solving skills through practice and critical thinking.

Understanding of the definition, properties, and applications of the isomorphism theorems in algebraic structures, as well as their significance in abstract algebra and beyond. The plan should be comprehensive and balanced in terms of theoretical concepts and practical problem solving exercises, enabling students to comprehend complex concepts and apply them effectively to problem solving.

III. Teaching content:

1. ***First Isomorphism Theorem*** If $\phi : G \to H$ is a group homomorphism with $Ker\phi$, then $K = Ker\phi$ is normal in G. Let $\psi : G \to G/K$ be the natural homomorphism. There exist an isomorphism

$$\eta : G/K \longrightarrow \phi(G)$$

such that $\phi = \eta\psi$.

2. ***Second Isomorphism Theorem*** Let H be a subgroup of a group G (not necessarily normal in G) and N be a normal subgroup of G. Then HN is a subgroup of G, $H \cap N$ is a normal subgroup of H, and

$$H/(H \cap N) \cong HN/N.$$

3. ***Correspondence Theorem*** Let N be a normal subgroup of a group G. There is a one to one correspondence between subgroups of G containing N and

subgroups of G/N,

$$\{ \text{ subgroups of G containing } N\} \longleftrightarrow \{\text{subgroups of } G/N\}.$$

Furthermore, the normal subgroups of G containing N correspond to the normal subgroup of G/N.

4. ***Third Isomorphism Theorem for groups*** Let G be a group and N and H be normal subgroups of G with $N \subseteq H$. Then

$$G/H \cong \frac{G/N}{H/N}.$$

IV. Teaching time:

3 classes.

3.6 Endmorphisms and automorphisms of groups

I. Teaching objective:

The teaching of group automorphisms is a key aspect of the study of abstract algebra and plays a vital role in understanding the structure and properties of groups. The primary objective of teaching group automorphisms is to enable students to develop a strong understanding of the concept of automorphism, and to use this knowledge to analyze the properties and structure of groups.

Students should be introduced to the definition and properties of an automorphism, along with concrete examples of group automorphisms. Students should learn about the relationship between group automorphisms and isomorphisms, and how group automorphisms can be used to classify groups.

II. Teaching plan:

The teaching should include a detailed discussion of the significance of group automorphisms in the study of abstract algebra, including their applications in Galois theory and the classification of groups. The emphasis should be on developing students' problem-solving skills through practice and critical thinking.

Group automorphisms should provide students with a deep and comprehensive understanding of the concept, properties, and applications of group automorphisms in abstract algebra. Engaging students in the study of abstract algebra and preparing them for future studies in this and other fields.

III. Teaching content:

1. An ***endmorphism*** of a group G is an homomorphism from G to itself. An ***automorphism*** of a group G is an isomorphism from G to itself.

2. If G is a group, then $Aut(G)$ is a subgroup of the symmetric group S_G.

3. Let G be a group. Then $Inn(G)$ is normal subgroup of $Aut(G)$.

4. Let G be a group, c is called a **center element** if for any element $x \in G$, $cx = xc$. Denote the set of center elements of G as $Center(G)$. Applying the First Isomorphism Theorem to the homomorphism δ. We will discover some interesting conclusion about inner automorphism and center of group in the following proposition.

5. Let G be a group and $\delta : G \to Aut(G)$ be the conjugation homomorphism. Then Ker(δ) = Center(G) and Im(δ) = $Inn(G)$. Hence $Inn(G) \cong G/Center(G)$.

IV. Teaching time:

2 class.

Chapter 4

Rings and Fields

4.1 Rings

I. Teaching objective:

Ring is an essential part of the study of abstract algebra and plays a vital role in fostering critical thinking and problem-solving abilities among students. The primary objective of teaching rings is to enable students to understand the concepts and properties of rings and develop an appreciation for the abstract structures involved. The basic concepts of abstract algebra, including groups and fields will be studied. Students should learn about the properties and axioms of rings, including the two binary operations of addition and multiplication, distributivity, associativity, and commutativity.

The teaching plan should focus on the different types of rings, including commutative, non-commutative, and other special types of rings, such as integral domains, fields, and division rings.

II. Teaching plan:

Teaching should include sufficient examples and problem solving exercises designed to foster critical thinking, reasoning abilities, and logical problem-solving skills in students. These exercises should range from basic concept based problems to application based advanced problems that require the application of different concepts. The teaching for rings should provide students with a comprehensive understanding of the concepts, properties, and applications of rings in abstract algebra. The plan should be balanced in terms of theoretical concepts and practical problem solving exercises, engaging students in the study of abstract algebra and preparing them for future studies in this and other related fields.

III. Teaching content:

1. If a non-empty set R has two closed binary operations, addition and multiplication, satisfying the following conditions, for $a, b, c \in R$,

(1) $a + b = b + a$

(2) $(a + b) + c = a + (b + c)$

(3) There is an element $0 \in R$ such that $0 + a = a$.

(4) There exists an element $-a \in R$ such that $a + (-a) = 0$.

(5) $(ab)c = a(bc)$.

(6) $a(b + c) = ab + ac; (a + b)c = ac + bc$.

2. A nonzero element $a \in R$ is called a ***zero divisor*** if there is a nonzero element b, such that $ab = 0$.

3. A commutative ring with identity is called an ***integral domain*** if for any $a, b \in R$ such that $ab = 0$, either $a = 0$ or $b = 0$.

4. A commutative divisor ring is called a ***field***.

5. Let D be a commutative ring with identity. The D is an integral domain if and only if for all nonzero elements a $\in$ D with ab=ac, we have b=c.

6. Every finite integral domain is a field.

7. The ***characteristic*** of a ring R is the least positive integer n such that $nr = 0$, for any $r \in R$. Denote as $char(R)$. If no such integer exists, then the characteristic of R is defined to be 0.

8. The characteristic of an integral domain is either prime or zero.

IV. Teaching time:

3 classes.

4.2 Subrings and Ideals

I. Teaching objective:

Teaching about subrings and ideals is a critical aspect of abstract algebra that helps students develop a more comprehensive understanding of rings and their properties. The primary objective of teaching subrings and ideals is to enable students to understand how to identify these structures and understand their significance in ring theory.

The teaching for subrings and ideals should begin with an introduction to the concepts of subrings and ideals, including their definitions, properties, and related mathematical concepts. The teaching plan should focus on examples and applications of subrings and ideals, including their significance in modern algebra, number theory, and algebraic geometry. Different types of problems, including applications of subring and ideal properties, should be incorporated into the teaching plan to provide a real-world perspective on the significance of these definitions in modern mathematics.

II. Teaching plan:

Teaching plan for subrings and ideals should provide students with a comprehensive understanding of these abstract algebra structures and their relationship to other mathematical concepts. It should encourage students to develop a deep fascination with mathematics, apply their knowledge to solve problems, and prepare them for future studies in this and other related fields. Discuss the relationship between subrings and ideals to other mathematical concepts, including modules and fields. This will help students to see the broader mathematical connections between algebraic structures and begin to develop a more holistic view of mathematical research and discovery.

III. Teaching content:

1. Let $(R, +, \cdot)$ be a ring. A ***subring*** S of R is a subset S of R such that $(S, +, \cdot)$ is a ring.

2. Let R be a ring and S a subset of R. Then S is a subring of R if and only if the following conditions are satisfied.

(1) $S \neq \emptyset$.

(2) $r - s \in S$ for all $r, s \in S$.

(3) $rs \in S$ for all $r, s \in S$.

3. A subring I of R is an ***ideal*** if $ar, ra \in I$, for $a \in I, r \in R$, or equivalent $rI \subseteq I, Ir \subseteq I$.

4. Let R be a ring, $a \in R$, an ideal of the form

$$\langle a \rangle = \{x_1 a y_1 + x_2 a y_2 + \cdots + x_m a y_m + sa + at + na | x_i, y_i, s, t \in R, n \in \mathbb{Z}\}$$

is called a ***principal ideal***.

5. There are only two ideals in a divisor ring, i.e., trivial ideals.

6. Every ideal in the ring of integers $\mathbb{Z}$ is a principle ideal.

7. Let F be a field. Then every ideal in $F[x]$ is a principal ideal.

IV. Teaching time:

3 classes.

4.3 Ring homomorphisms

I. Teaching objective:

The objective of teaching ring homomorphisms is to provide an in-depth understanding of the concept and its applications in various fields of mathematics. The students must be able to understand the relationship between rings and homomorphisms and use this knowledge to solve problems in algebra and geometry.

II. Teaching plan:

The teaching plan for ring homomorphisms should begin with an introduction to the concept, its definition, properties, and examples. The students must be able to understand the concept of ring homomorphisms and its relationship with rings. The next part of the teaching plan should focus on the basic properties of ring homomorphisms, such as injectivity, surjectivity, and bijectivity. The students must be able to understand the significance of these properties and how they apply to the study of ring homomorphisms. The third part of the teaching plan should focus on the applications of ring homomorphisms in algebra and geometry. The students must be able to use their knowledge of ring homomorphisms to solve problems related to group theory, abstract algebra, and geometry.

The teaching plan should conclude with a review of the concepts covered in the course, including an assessment of the student's ability to apply their knowledge to solve problems. The students must be able to demonstrate their understanding of the concept of ring homomorphisms, their properties, and their applications in various fields of mathematics.

III. Teaching content:

1. Let R and S be rings, a map $\phi : R \longrightarrow S$ is a ***ring homomorphism***, if for all $a, b \in R$,

$$\begin{aligned} \phi(a+b) &= \phi(a)+\phi(b), \\ \phi(ab) &= \phi(a)\phi(b). \end{aligned}$$

2. The ***kernel*** of a ring homomorphism to be the set

$$\mathrm{Ker}\phi = \{a \mid \phi(a) = 0, a \in R\}. \tag{4.1}$$

3. The kernel of any ring homomorphism $\phi : R \to S$ is an ideal in R.

4. Let I be an ideal of R. The factor group R/I is a ring with multiplication defined by

$$(r+I)(s+I) = rs + I. \tag{4.2}$$

R/I is called a ***factor ring*** or ***quotient ring***.

5. The map $\psi : R \to R/I$ is called ***natural homomorphism*** or ***canonical homomorphism.***

6. ***First Isomorphism Theorem for Rings*** Let $\phi : R \to S$ be a ring homomorphism. Then $\mathrm{Ker}\phi$ is an ideal of R. If $\psi : R \to R/\ker\phi$ is the canonical homomorphism, then there exists an isomorphism

$$\eta : R/\mathrm{Ker}\phi \to \phi(\mathrm{R})$$

such that $\phi = \eta\psi$.

7. ***Second Isomorphism Theorem for rings*** Let I be a subring of a ring R and J an ideal of R. Then $I \cap J$ is an ideal of I, and

$$I/I \cap J \cong (I+J)/J. \tag{4.3}$$

8. ***Third Isomorphism Theorem*** Let R be a ring and I and J be ideals of R where $J \subset I$. Then

$$R/I \cong \frac{R/J}{I/J}. \tag{4.4}$$

IV. Teaching time:

2 classes.

4.4 Maximal ideals and prime ideals

I. Teaching objective:

To provide an understanding of the concepts and maximal and prime ideals' importance in the study of abstract algebra. The students must be able to recognize the properties and relationships between maximal and prime ideals and use this knowledge to solve problems in algebra.

II. Teaching plan:

eaching Plan for Maximal and Prime Ideals:

Teachers should begin with an introduction to the concept of ideals, their definition, and their properties. The students must be able to understand the relationship between ideals and rings and how they apply to the study of algebra. Teachers should focus on the definition of maximal ideals, their properties, and their relationship with prime ideals. The students must be able to understand the significance of these concepts and how they can use them to solve problems in algebra.

The applications of maximal and prime ideals in algebra and geometry is importan. The students should be able to use their knowledge of maximal and prime ideals to study the structure of rings, modules, and fields, including how they relate to concepts such as quotient rings, localization, and field extensions.

III. Teaching content:

1. A proper ideal M of a ring R is a ***maximal ideal*** of R if the ideal M is not a proper subset of any ideal of R except R itself.

2. Let n be a positive integer. Then $\langle n \rangle$ is a maximal ideal of $\mathbb{Z}$ if and only if n is prime.

3. Let R be a commutative ring with identity and M an ideal in R. Then M is a maximal ideal of R if and only if R/M is a field.

4. A proper ideal P in a commutative ring R is called a ***prime ideal*** if whenever $ab \in P$, then either $a \in P$ or $b \in P$.

5. Let n be a positive integer. Then $\langle n \rangle$ is a prime ideal of $\mathbb{Z}$ if and only if n is prime.

6. Let F be a field and suppose that $p(x) \in F[x]$. Then the ideal generated by $p(x)$ is maximal if and only if $p(x)$ is irreducible.

IV. Teaching time:

3 classes.

4.5 Extension fields

I. Teaching objective:

The objective of teaching field extensions is to provide an understanding of the concepts and their importance in the study of abstract algebra. The students must be able to recognize the properties and relationships between field extensions and use this knowledge to solve problems in algebra.

II. Teaching plan:

The teaching plan for field extensions should begin with an introduction to the concept of fields, their definition, and their properties. The students must be able to understand the relationship between fields, and how they apply to the study of algebra. The students must be able to understand the significance of these concepts and how they can use them to solve problems in algebra. The students should be able to use their knowledge of field extensions to apply it to study the structure of various algebraic objects, including polynomial rings, domains, and other algebraic structures. The students must be able to demonstrate their understanding of the properties and relationships between the field extensions and their parents, autonomously concretizing abstract algebraic notio

III. Teaching content:

1. Let E be a field. A ***subfield*** F is a subset of E and F is a field. A field E is an ***extension field*** of a field F if F is a subfield of E. The field F is called

the ***base field***. We write $F \subset E$ or E/F.

2. If an extension field E of a field F is a finite dimensional vector space over F of dimension n, then we say that E is a finite extension of ***degree*** n over F. We write $[E : F] = n$ to indicate the dimension of E over F.

3. If E is a field extension of F and $\alpha_1, \cdots, \alpha_n$ are contained in E, we denote the smallest extension containing F and $\alpha_1, \cdots, \alpha_n$ by $F(\alpha_1, \cdots, \alpha_n)$. If $E = F(\alpha)$ for some $\alpha \in E$, then E is a ***simple extension*** of F.

4. ***Telescope Formula*** If E is a finite extension of F and K is a finite extension of E, then K is a finite extension of F and

$$[K : F] = [K : E][E : F]. \tag{4.5}$$

IV. Teaching time:

2 classes.

4.6 Algebraic extension

I. Teaching objective:

Algebraic extensions is to provide the students with an in-depth understanding of algebraic structures and their properties. The students must be able to recognize and apply the concepts of algebraic extensions to solve complex mathematical problems. The students must demonstrate their competency in applying their understanding of algebraic extensions, using concrete examples to consolidate abstract algebraic notions.

II. Teaching plan:

Teachers should focus on the definition and properties of algebraic extensions and how they relate to field extensions. The students must understand how to determine whether a given field extension is an algebraic extension and how to construct algebraic extensions that satisfy specific properties. The students should understand the significance of theorems in this part and how to use them to solve problems in algebraic extensions.

III. Teaching content:

1. An element α in an extension field E over F is ***algebraic*** over F if

$f(\alpha) = 0$ for some nonzero polynomial $f(x) \in F[x]$. An element in E that is not algebraic over F is ***transcendental*** over F. An extension field E of a field F is an ***algebraic extension*** of F if every element in E is algebraic over F.

2. A field extension of finite degree is algebraic.

3. Let E be an extension field of a field F and $\alpha \in E$ with α algebraic over F . Then there is a unique irreducible monic polynomial $p(x) \in F(x)$ of smallest degree such that $p(\alpha) = 0$. If $f(x)$ is another polynomial in $F[x]$ such that $f(\alpha) = 0$, then $p(x)$ divides $f(x)$.

4. Let E be an extension field of F and $\alpha \in E$ be algebraic over F. The unique monic polynomial $p(x)$ of the last theorem is called the ***minimal polynomial*** for α over F. The degree of $p(x)$ is the degree of α over F.

5. If α is algebraic over F, then the degree of $F(\alpha)$ over F is equal to the degree of the minimum polynomial of α over F.

6. The algebraic numbers form a subfield of $\mathbb{C}$.

7. Let E be a field extension of a field F. We define the **algebraic closure** of a field F in E to be the field consisting of all elements in E that are algebraic over F. A field F is **algebraically closed** if every nonconstant polynomial in $F[x]$ has a root in F.

8. A field F is algebraically closed if and only if every nonconstant polynomial in $F[x]$ factors into factors with degree 1 over $F[x]$.

IV. Teaching time:

2 classes.