Workbook

Chapter 1. Sets

1. Let A, B and C be sets. Show that

   (1) $(A \cap B) \setminus B = \phi$.

   (2) $(A \cup B) \setminus B = A \setminus B$.

   (3) $(B \cup C) \setminus A = (B \setminus A) \cup (C \setminus A)$.

   (4) $(B \cap C) \setminus A = (B \setminus A) \cap (C \setminus A)$.

2. Let A, B, C be sets. Prove that

   (1) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

   (2) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

3. Let X, Y be finite sets. Prove that

$$|X \cup Y| + |X \cap Y| = |X| + |Y|.$$

4. Let $X = \{\emptyset\}$. What is $P(X)$?

5. Write down the power sets of $A = \{a, b, c, d\}$ and $B = \{\phi, \{\phi\}\}$.

6. Show that $f : \mathbb{R}^+ \longrightarrow \mathbb{R} : x \longmapsto log_{10}(x)$ is a bijection.

7. Define the order of a set A is the number of elements in A, denote as $|A|$. How many maps from A to B if $|A| = n$ and $|B| = m$?

8. Show that a map is invertible if and only if it is both injective and surjective.

9. Let $A = \{0, 1, 2, 3, \cdots\}$, $B = \{1, 2, 3, \cdots\}$. Prove that $A \cong B$.

10. Let X be a set, A be a subset of X. Is $f : P(X) \to P(X) : A \mapsto A'$ a bijection? Why?

11. Is the relation $aRb \Leftrightarrow ab > 0$ for $a, b \in \mathbb{R}$ an equivalent relation? Why?

12. Define a relation R on $\mathbb{R}^2$ by stating that $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 \leq c^2 + d^2$. Show that $\sim$ is reflexive and transitive, but it is not symmetric.

13. Give all partitions on $A = \{1, 2, 3, 4\}$.

14. Write the addition sum of all elements in $\mathbb{Z}_7$.

15. Write the multiplication product of all elements in $\mathbb{Z}_8$.

16. Write down the tables of $(\mathbb{Z}_7, +)$ and $(\mathbb{Z}_7, \cdot)$.

17. Let X be a set and R is a relation of X. Define xRy if $x|y$. Is R an equivalence relation, partial ordering relation or totally ordering relation?

18. Let a be a nonzero integer and $n \neq 0$ be a natural number. Then $gcd(a, n) = 1$ if and only if there exists a multiplication inverse b for $a(mod\ n)$; that is, a nonzero integer b such that $ab \equiv 1(mod\ n)$.

19. Let a and b be nonzero integers. Then there exist integers r and s such that $gcd(a, b) = ar + bs$.

20. Calculate $gcd(a, b)$ and find integers r and s such that $gcd(a, b) = ar + bs$.

    (1) a=15, b=26; (2) a=165, b=234; (3) a=48, b=120.

21. Define the **least common multiple** of two nonzero integers a and b, denoted by $lcm(a, b)$, to be the nonnegative integer m such that both a and b divide m, and if a and b divide any other integer n, then m also divides n. Prove there exists a unique least common multiple for any two integers a and b.

22. Show that $lcm(a, b) = ab$ if and only if $gcd(a, b) = 1$

23. If $d = gcd(a, b)$ and $m = lcm(a, b)$, prove that $dm = |ab|$.

Chapter 2. Groups

1. Define $a \circ b = a + b + ab$ for $\forall a, b \in \mathbb{N}$. Is $(\mathbb{N}, \circ)$ a group?

2. Show that $(\mathbb{Z}_n, +)$ is a group, but $(\mathbb{Z}_n, \cdot)$ is not a group.

3. Let $(G, \cdot)$ ba a group, then G is an abelian group if and only if $\forall a, b \in G, (ab)^2 = a^2b^2$.

4. Let G be a finite group, $\forall g \in G$, then the order of g is finite.

5. Let G be a group with order $|G| = n$. S is a subset of G, with $|S| > \frac{n}{2}$. Show that $\forall g \in G$, there exist $a, b \in S$ such that $g = ab$.

6. . Let a, b be two elements of a group G, and $aba = ba^2b, a^3 = 1, b^{2n-1} = 1$. Then $b = 1$.

7. Show that the intersection of two subgroups of a group G is a subgroup of G.

8. Let H be a subgroup of G, If $g \in G$, show that

$$gHg^{-1} = \{g^{-1}hg | h \in H\}$$

   is also a subgroup of G.

9. Prove pr disprove : If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

10. Let G be a group and $g \in G$. Show that the center of G, $Z(G) = \{x \in G | gx = xg, \forall g \in G\}$ is a subgroup of G.

11. Let H be a subgroup of G and $C(H) = \{g \in G | gh = hg, \forall h \in H\}$. Prove $C(H)$ is a subgroup of G **centralizer** of H in G.

12. Give the order of every element in A_4.

13. Let $\tau = (1\ 2\ 3\ 4\ 5\ 6)$. What is $\langle \tau \rangle$.

14. Find out all subgroup of S_3.

15. Write down the order of every element in $\mathbb{Z}_7$ and $\mathbb{Z}_{10}$. A:

$$\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$$

$|\overline{0}| = 1, |\overline{1}| = 7, |\overline{2}| = 7, |\overline{3}| = 7, |\overline{4}| = 7, |\overline{5}| = 7, |\overline{6}| = 7.$

16. Find out all subgroups of $\mathbb{Z}_8$. What are cyclic subgroups of $\mathbb{Z}_8$?

17. Let G be a group, $a, b \in G$, and $ab = ba$, $|a| = m, |b| = n$, $gcd(m, n) = 1$. Show that $|ab| = mn$.

18. Show that the group with prime order is a cyclic group.

19. Let G be a group, g in G, $|g| = mn$, and $(m, n) = 1$, then $g = ab$ where $|a| = m, |b| = n$, and $a, b \in G$.

20. Let a, b be elements of a group G such that $a^3 = b^2 = e, (ab)^2 = e, a^2 \neq e, b \neq e$. What is $< a, b >$.

21. Let a, b be elements of a group G such that $a^3 = b^2 = e, ab = ba, a^2 \neq e, b \neq e$. What is $< a, b >$

22. Let G be a finite abelian group. Prove that the product of all the elements of G equals the product of all the elements of G of order 2.

23. (Wilson's Theorem) If p is a prime, then $(p-1)! = -1(\mod p)$.

24. Find out all permutations of the set $X = \{a, b, c, d\}$.

25. Computing the product of permutation: $(124)(234), (3124)(4561)$.

26. Please write $(456)(567)(761)$ as product of transpositions.

27. What is S_4?

28. Let G be a group and define a map $\lambda_g : G \longrightarrow G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G.

29. Write down the dihedral group of D_5.

30. Show that the order of cycle $(i_1 i_2 \cdots i_k)$ is k.

Chapter 3. Properties of. groups

1. Find out all subgroups of A_4 with order 2 and 3.

2. Suppose that G is a finite group with 60 elements. What are the orders of possible subgroups of G?

3. List the left and right cosets of subgroup $\langle 3 \rangle$ in group $U(8)$.

4. Suppose that $[G : H] = 2$. If a and b are not in H, show that $ab \in H$.

5. Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. What are left cosets of H in A_4.

6. What are right cosets of $H =< \overline{4} >$ in $\mathbb{Z}_{12}$.

7. Let H and K be subgroups of G, and $|H| = 12, |K| = 35$. What is $H \cap K$?

8. Let $n \neq 0$. Prove that $\mathbb{Z} \cong n\mathbb{Z}$

9. Prove or disprove $U(8) \cong \mathbb{Z}_4$.

10. Prove S_4 is not isomorphic to D_{12}.

11. Let $G = \{(a, b)|a, b \in \mathbb{R}, a \neq 0\}$ with $(a, b)(c, d) = (ac, ad + b)$ be a group, $K = \{(1, b)|b \in \mathbb{R}\}$. Show that $G/K \cong \mathbb{R}^*$.

12. Define the center of a group G to be the set $C(G) = \{x \in G|xg = gx, \forall g \in G\}$. Show that $C(G)$ is a normal subgroup of G.

13. Let $\phi : \mathbb{Z} \to \mathbb{Z} : m \mapsto 7m$. Prove that ϕ is a group homomorphism. Find the kernel and the image of ϕ.

14. Find out all possible homomorphism from $\mathbb{Z}_7 \longrightarrow \mathbb{Z}_{12}$.

15. Let A be an $m \times n$ matrix. Show that map

$$\phi : \mathbb{R}^n \to \mathbb{R}^m : \alpha \mapsto A\alpha$$

is a homomorphism.

16. Let $G = \{(a,b) | a, b \in \mathbb{R}, a \neq 0\}$ with $(a,b)(c,d) = (ac, ad+b)$ be a group, $K = \{(1,b) | b \in \mathbb{R}\}$. Show that $G/K \cong \mathbb{R}^*$.

17. In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$. What is $H + N$ and $H \cap N$.

18. What is the $Aut(\mathbb{Z}_8)$? Is $Aut(\mathbb{Z}_8)$ a cyclic group?

19. Let $K_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$. What is $Aut(K_4)$?

20. Let $G = \{e, g, g^2, g^3\}$. What is $Aut(G)$?

21. Let $G = \{e, g, g^2, \cdots, g^{n-1}\}$. What is $Aut(G)$?

Chapter 4. Rings

1. Show that $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$ is a domain.

2. Find out all zero divisors of $\mathbb{Z}_6$.

3. Prove that $R = \{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} |, x, y \in \mathbb{R}\}$ is a ring.

4. What are zero divisor of ring $R = \{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} |, x, y \in \mathbb{R}\}$.

5. Let p be a prime. Prove that $(\mathbb{Z}_p, +, \cdot)$ is a field.

6. Let I and J be ideals of ring R.

   (1) Prove that $I \cap J$ is an ideal of R.

   (2) Is $I \cup J$ an ideal?

7. Let I and J be ideals of ring R. Define $IJ = \{\sum_i a_i b_i | a_i \in I, b_i \in J\}$. Prove that IJ is an ideal of R

8. Let I and J be ideals of ring R. Define $I + J = \{a + b | a \in I, b \in J\}$. Prove that $I + J$ is an ideal of R

9. Let I be an ideal of R. Define $r(I) = \{r \in R | ru = 0, \forall u \in U\}$. Prove that $r(I)$ is an ideal of R.

10. Let $a, b \in \mathbb{Z}$. What is $< a, b >$?

11. Find out all ring homomorphisms of $\mathbb{Z}_{12}$ to $\mathbb{Z}_6$.

12. Find out all prime ideals of $\mathbb{Z}_{18}$.

13. Let $CharR = p$, p be a prime. Then $(a + b)^p = a^p + b^p, a, b \in \mathbb{R}$.

14. Find out all prime ideals and maximal ideal of $\mathbb{Z}_{16}$.

15. Let

$$R = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, c, d \in \mathbb{R} \right\}$$

and

$$I = \left\{ \begin{pmatrix} r & 0 \\ s & 0 \end{pmatrix} \mid r, s \in \mathbb{R} \right\}.$$

Then R is a ring under addition and multiplicartion of matrix. Moreover, I is a maximal ideal of R.

16. Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$.

17. Let $a, b \in \mathbb{R}, b \neq 0$. Show that $\mathbb{R}(a+bi) = \mathbb{C}$.

18. Prove or disprove that $\mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{-3})$.

19. Prove that $\mathbb{Q}(4-i) = \mathbb{Q}(1+i)$.

20. What is the quotient field of Z.

21. Find out algebraic elements.

    (1) $\sqrt{24}$; (2) $\sqrt{13}$; (3) e^2; (4) $\pi + 6$; (5) $\pi - e$.

22. Show that a is algebraic over F if and only if a^2 is algebraic over F.

23. Show that $\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}/\langle x^3 - 2 \rangle$.

24. Let $\frac{1}{1+2\sqrt[3]{2}+3\sqrt[3]{4}} = a + b\sqrt[3]{2} + c\sqrt[3]{4}$, $a, b, c \in \mathbb{Q}$. Compute a, b, c.

25. Give the minimal polynomial over $\mathbb{Q}$ of these elements:

    (1) $\sqrt{2} + \sqrt{-3}$; (2) $\sqrt[3]{3} + \sqrt{3}$; (3) $\sqrt[3]{2} - \sqrt[3]{4}$.