实物期权与企业投资决策

第九讲 连续时间模型

Model of the behavior of stock prices

- Wiener processes

Definition: A *Wiener (Gaussian) process* or *Brownian motion process* $\{z(t), t \in [0, \infty)\}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ which satisfies:

1. zero initial value: $z(0) = 0$ a.s.
2. the Markov property: the probability distributions for any future $z(t+s)$ depend only on the current value $z(t)$, and are not affected by its past history.
3. independent increments: increments (changes), $z(t_{i+1}) - z(t_i)$, in the process over any finite time interval are independent of those over any other nonoverlapping time interval, $z(t_{j+1}) - z(t_j)$, $(t_{i+1} - t_i) \cap$

$z(t_{j+1}) - z(t_j) \cap z(t_{i+1}) - z(t_i) = \phi$

4. normally-distributed increments: increments (changes), $z(t+s)-z(t)$, follow

$$z(t+s)-z(t)\sim N\left(0,\sigma^2 s\right), \quad (3)$$

where σ is a constant.

5. For each $\omega\in\Omega$, $z(t;\omega)$ is continuous in $t\in[0,\infty)$.

Note:

- When $\sigma=1$ in (3), z is said to be a *standard* Wiener process (Brownian motion). In what follows, z represents a standard Wiener process; $z(t+s)-z(t)\sim N(0,s)$.
- A nonstandard Wiener process y, say $y(t+s)-y(t)\sim N(0,\sigma^2 s)$, can be represented in terms of a standard Wiener process z as

$$y(t)=\sigma z(t).$$

- For a standard wiener process z,

(a) $E[dz(t)]=0,$

(b) $E\left[(dz(t))^2\right]=dt,$

(c) $E\left[(dz(t))^{2+n}\right]=0, n=1,2,\cdots,$

(d) $E[dz(t)\,dz(s)]=0, t\neq s,$

(e) $(dz(t))^2=dt,$

(f) $dz(t)dt=0.$

3. For two standard Wiener processes z and w with the instantaneous correlation coefficient of $dz(t)$ and $dw(t)$ equal to ρ,

(a) $\mathrm{Cov}_t(dz(t), dw(t)) = \rho dt$,

(b) $dz(t)dw(t) = \rho dt$.

	dz	dw	dt
dz	dt	ρdt	0
dw		dt	0
dt			0

- The process for stock prices
  - Usually we assume the process of stock prices as geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz$$

the discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

$\Delta S/S$ is normally distribution

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma \sqrt{\Delta t})$$

  - The limit of binomial tree process is Geometric Brownian motion as $\Delta t \to 0$.

- Itos lemma

Let F be a function of Ito process p and time t, $F(p,t)$, where

$$\mathrm{d}p(t) = \mu(t, p(t))\,\mathrm{d}t + \sigma(t, p(t))\,\mathrm{d}z(t),$$

with the required technical conditions being satisfied. And assume:

(a) $F(p,t)$ is at least twice differentiable w.r.t. p;

(b) $F(p,t)$ is at least once differentiable w.r.t. t.

Then, instantaneous increments of F are given by

$$\begin{aligned}\mathrm{d}F &= \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial p}\mathrm{d}p + \frac{1}{2}\frac{\partial^2 F}{\partial p^2}(\mathrm{d}p)^2 \\ &= \left\{\frac{\partial F}{\partial t} + \eta\frac{\partial F}{\partial p} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial p^2}\right\}\mathrm{d}t + \sigma\frac{\partial F}{\partial p}\mathrm{d}z.\end{aligned}$$

- Example
  - Lognormal property of stock prices
    - If a stock price, S, follows Geometric Brownian motion,

$$dS = \mu S dt + \sigma S dz$$

then,

$$d\ln S = (\mu - \sigma^2/2)dt + \sigma dz$$

- Derivation of the Black-Scholes model
  - Assumptions:
    - The stock price follows the process with μ and σ constant.
    - The short selling of securities is permitted.
    - There are no transactions costs or taxes. All securities are perfectly divisible.
    - There are no dividends during the life of the derivative.
    - There are no risk-less arbitrage opportunities.
    - Security trading is continuous.
    - The risk-free rate of interest, r, is constant and the same for all maturities.
  - Method 1: risk-free portfolio method.
  - Method 2: martingale approach.

Martingale

Definition: an adapted sequence $(M_n)_{0 \le n \le N}$ of real random variables is

- a martingale if $E(M_{n+1}|F_n) = M_n$ for all $n \le N-1$
- a supermartingale if $E(M_{n+1}|F_n) \le M_n$ for all $n \le N-1$
- a submartingale if $E(M_{n+1}|F_n) \ge M_n$ for all $n \le N-1$

1. $(M_n)_{0 \leq n \leq N}$ is a martingale if and only if $E(M_{n+j}|F_n) = M_n, \forall_j \geq 0$.

2. The sum of two martingale is martingale.

3. Obviously, similar properties can be shown for supermartingales and submartingales.

- Option on a stock paying a continuous dividend yield
  - Assume the dividend yield is q, the payment of a continuous dividend yield causes the growth rate in the stock price to be less than it would otherwise be by an amount q.
  - The probabilities distribution for the stock price at time T in each of the following two cases are equivalent:
    - The stock starts at price S_0 and pays a continuous dividend yield at rate q.
    - The stock starts at price S_0e^{-qT} and pays no dividend yield.

- Option pricing formula

$$
\begin{aligned}
c &= S_0 e^{-qT} N(d_1) - X e^{-rT} N(d_2) \\
p &= X e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \\
d_1 &= \frac{\ln(S_0/X) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

- The risk-neutral process for the stock price is

$$dS = (r - q)S dt + \sigma S dz$$

- Futures options
  - The expected gain to the holder of a futures contract in a risk-neutral world is zero.
  - Option pricing formula

$$
\begin{aligned}
c &= e^{-rT}[F_0 N(d_1) - X e^{-rT} N(d_2)] \\
p &= e^{-rT}[X N(-d_2) - F_0 N(-d_1)] \\
d_1 &= \frac{\ln(F_0/X) + \sigma^2 T/2}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

实物期权分析中项目净现值波动率的估计

- 估计影响项目净现值的各个因素(如：销售额、价格、成本等)的波动率。一般有两种方法：
  - 根据各因素的历史数据，采用计量模型估计其历史波动率。该方法的前提是：未来的波动率等于其历史波动率。
  - 在多数情况下，我们没有足够的历史数据。并且，未来的波动率很可能与过去相比有较大的偏差。此时，我们只好依赖管理者给出某一影响因素(如：价格等)的波动范围，然后根据这一波动范围估计其波动率。

根据管理者的估计范围计算波动率

- 假设某一风险因素(如：价格)遵从于几何布朗运动。

$$\frac{dV_{t+\Delta t}}{V_t} = \mu dt + \sigma dz_t$$

根据伊藤引理：

$$V_{t+\Delta t} = V_t e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon}$$

$$\varepsilon \sim N(0,1)$$

令$r_t = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon$

因此，$r_t \sim N(\overline{r}_t, \sigma\sqrt{\Delta t}), \overline{r}_t = (\mu - \frac{1}{2}\sigma^2)\Delta t$

当$T = n\Delta t$时，

$V_T = V_0 e^{\sum r_t}$，其95%置信区间为：

$$V_T^{上限} = V_0 e^{\sum \overline{r}_t + 2\sigma\sqrt{T}}, V_T^{下限} = V_0 e^{\sum \overline{r}_t - 2\sigma\sqrt{T}}$$

因此，

$$\sigma = \frac{\ln\left(\frac{V_T^{上限}}{V_0}\right) - \sum \overline{r}_t}{2\sqrt{T}}，或\sigma = \frac{\sum \overline{r}_t - \ln\left(\frac{V_T^{下限}}{V_0}\right)}{2\sqrt{T}}$$