Overview of Optimization Models

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Introduction

Optimization is the process of finding the best way of making decisions that satisfy a set of constraints.

An Optimization Model

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{X} \subseteq \mathbb{R}^n$.

- **x:**the vector of decision variables
- **f(x):**the objective function
- $\mathcal{X}$**:**the constraint set or feasible region

Introduction

The constraint set is often expressed in terms of equalities and inequalities involving additional functions. More precisely, the constraint set $\mathcal{X}$ is often of the form

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) = b_i, \text{ for } i = 1, \ldots, m, \text{ and } h_j(\mathbf{x}) \leq d_j, \text{ for } j = 1, \ldots, p\}$$

for some $g_i, h_j : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, p$. When this is the case, the optimization problem is usually written in the form

The Optimization Model

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) = b_i, \quad i = 1, \ldots, m \\ & h_j(\mathbf{x}) \leq d_j, \quad j = 1, \ldots, p \end{aligned}$$

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Types of Optimization Models

Particular structural assumptions on the objective and constraints of the problem give rise to different classes of optimization models with various degrees of computational difficulty. We should note that the following is only a partial classification based on the current generic tractability of various types of optimization models.

- **Convex optimization**: These are problems where the objective $f(\mathbf{x})$ is a convex function and the constraint set $\mathcal{X}$ is a convex set.
- **Mixed integer optimization**: These are problems where some of the variables are restricted to take integer values. This restriction makes the constraint set $\mathcal{X}$ non-convex.
- **Stochastic and dynamic optimization**: These are problems involving random and time-dependent features. This class of optimization models is tractable only in some special cases.

Types of Optimization Models

A good portion of the models will be convex optimization models due to their favorable mathematical and computational properties.

There are two special types of convex optimization problems that we will use particularly often: **linear and quadratic programming**.

We now present a high-level description of four major classes of optimization models:

- linear programming
- quadratic programming
- mixed integer programming
- stochastic optimization

Linear Programming

A linear programming model is an optimization problem where the objective is a linear function and the constraint set is defined by finitely many linear equalities and linear inequalities.

Model

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{D}\mathbf{x} \geq \mathbf{d} \end{aligned}$$

for some vectors $\mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m, \mathbf{d} \in \mathbb{R}^p$ and matrices $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{D} \in \mathbb{R}^{p\times n}$.

The two best known and most successful methods for solving linear programs are the **simplex method** and **interior-point methods**. We briefly discuss these algorithms in Chapter 2.

Quadratic Programming

Quadratic programming, also known as quadratic optimization, is an extension of linear programming where the objective function includes a quadratic term and constraints are linear.

Model

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{D}\mathbf{x} \geq \mathbf{d} \end{aligned}$$

for some vectors and matrices $\mathbf{Q} \in \mathbb{R}^{n\times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m, \mathbf{d} \in \mathbb{R}^p, \mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times n}$. It is customary to assume that the matrix $\mathbf{Q}$ is symmetric.

Quadratic Programming

This assumption can be made without loss of generality since

$$\mathbf{x}^\top \mathbf{Q} \mathbf{x} = \mathbf{x}^\top \tilde{\mathbf{Q}} \mathbf{x}$$

where $\tilde{\mathbf{Q}} = \frac{1}{2}\left(\mathbf{Q} + \mathbf{Q}^\top\right)$, which is clearly a symmetric matrix.

We note that a quadratic function $\frac{1}{2}\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{c}^\top \mathbf{x}$ is convex if and only if the matrix Q is positive semidefinite ($\mathrm{x}^\top \mathbf{Q} \mathbf{x} \geq 0$ for all $x \in \mathrm{R}^n$). In this case the above quadratic program is a convex optimization problem and can be solved efficiently.

The two best known methods for solving convex quadratic programs are **active-set methods** and **interior-point methods**. We briefly discuss these algorithms in Chapter 5.

Mixed Integer Programming

A mixed-integer program is an optimization problem that restricts some or all of the decision variables to take integer values.

Model

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathrm{c}^\top \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{D}\mathbf{x} \geq \mathbf{d} \\
& x_j \in \mathbb{Z}, j \in J
\end{aligned}
$$

for some vectors and matrices
$\mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^{nn}, \mathbf{d} \in \mathbb{R}^p, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{D} \in \mathbb{R}^{p \times n}$ and some $J \subseteq \{1, \ldots, n\}$.

An important case occurs when the model includes binary variables, that is, variables that are restricted to take values 0 or 1.

The main classes of methods for solving mixed integer programs are **branch and bound**, **cutting planes**, and **a combination of these two approaches known as branch and cut**. We briefly discuss these algorithms in Chapter 8.

Stochastic optimization models are optimization problems that account for randomness in their objective or constraints.

Model

$$\min_{\mathbf{x}} \quad \mathbb{E}(F(\mathbf{x}, \omega))$$
$$\mathbf{x} \in \mathcal{X}.$$

In this problem the set of decisions $\mathbf{x}$ must be made before a random outcome ω occurs. The goal is to optimize the expectation of some function that depends on both the decision vector $\mathbf{x}$ and the random outcome ω.

Stochastic Optimization

A variation of this formulation, that has led to important developments, is to replace the expectation by some kind of risk measure ϱ in the objective:

Model

$$\min_{\mathbf{x}} \quad \varrho(F(\mathbf{x}, \omega))$$
$$\mathrm{x} \in \mathcal{X}.$$

There are numerous refinements and variants of the above two formulations. In particular, the class of two-stage stochastic optimization with recourse has been widely studied in the stochastic programming community. In this setting a set of decisions x must be made in stage one. Between stage one and stage two a random outcome ω occurs. At stage two we have the opportunity to make some second-stage recourse decisions $\mathbf{y}(\omega)$ that may depend on the random outcome ω.

Stochastic Optimization

The two-stage stochastic optimization problem with recourse can be formally stated as

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) + \mathbb{E}[Q(\mathbf{x}, \omega)] \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

The recourse term $Q(\mathbf{x}, \omega)$ depends on the first-stage decisions $\mathbf{x}$ and the random outcome ω. It is of the form

$$\begin{aligned} Q(\mathbf{x}, \omega) := \min_{\mathbf{y}(\omega)} \quad & g(\mathbf{y}(\omega), \omega) \\ & \mathbf{y}(\omega) \in \mathcal{Y}(\mathbf{x}, \omega) \end{aligned}$$

The second-stage decisions $\mathbf{y}(\omega)$ are adaptive to the random outcome ω because they are made after ω is revealed. The objective function in a two-stage stochastic optimization problem contains a term for the stage-one decisions and a term for the stage-two decisions where the latter term involves an expectation over the random outcomes.

Solution to Optimization Problems

The solution to an optimization problem can often be characterized in terms of a set of optimality conditions. Optimality conditions are derived from the mathematical relationship between the objective and constraints in the problem. In special cases, these optimality conditions can be solved analytically and used to infer properties about the optimal solution. However, in many cases, we rely on numerical solvers to obtain the solution to the optimization models.

Throughout this book we will illustrate examples with two popular solvers, namely Excel Solver and the MATLAB-based optimization modeling framework CVX.

Solution to Optimization Problems

There are far more sophisticated solvers such as the commercial solvers IBM $^{8}-$ ILOG $^{(3)}$ CPLEX $^{(3)}$, Gurobi, FICO $^{(3)}$ Xpress, and the ones available via the open-source projects COIN-OR or SCIP.

Optimization problems can be formulated using modeling languages such as AMPL, GAMS, MOSEL, or OPL. The need for these modeling languages arises when the size of the formulation is large. A modeling language lets people use common notation and familiar concepts to formulate optimization models and examine solutions. Most importantly, large problems can be formulated in a compact way. Once the problem has been formulated using a modeling language, it can be solved using any number of solvers.

Financial Optimization Models

We will focus on the use of optimization models for financial problems such as **portfolio management**, **risk management**, **asset and liability management**,**trade execution** , and **dynamic asset management**.

Optimization models are also widely used in other areas of business, science, and engineering, but this will not be the subject of our discussion.

Portfolio Management

One of the best known optimization models in finance is the portfolio selection model of Markowitz (1952). The gist of this model is to formalize the principle of diversification when selecting a portfolio in a universe of risky assets. As we discuss in detail in Chapter 6, Markowitz's mean–variance model and a wide range of its variations can be stated as a quadratic programming problem of the form

Model

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\gamma \cdot \mathbf{x}^\top \mathbf{V}\mathbf{x} - \boldsymbol{\mu}^\top \mathbf{x} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{D}\mathbf{x} \geq \mathbf{d}. \end{aligned} \tag{1.3}$$

Portfolio Management

The vector of decision variables $\mathbf{x}$ in model (1.3) represents the portfolio holdings. These holdings typically represent the percentages invested in each asset and thus are often subject to the full investment constraint $\mathbf{1}^\top \mathbf{x} = 1$. Other common constraints include the long-only constraint $\mathbf{x} \geq \mathbf{0}$, as well as restrictions related to sector or industry composition, turnover, etc. The terms $\mathbf{x}^\top \mathbf{V} \mathbf{x}$ and $\boldsymbol{\mu}^\top \mathbf{x}$ in the objective function are respectively the variance, which is a measure of risk,and the expected return of the portfolio defined by $\mathbf{x}$. The risk-aversion constant $\gamma > 0$ in the objective determines the tradeoff between risk and return of the portfolio.

Risk Management

Risk is inherent in most economic activities. Since companies cannot usually insure themselves completely against risk, they have to manage it. Poor risk management led to several spectacular failures in the financial industry in the 1990s. The modeling of regulatory constraints as well as other risk-related constraints that the firm wishes to impose to prevent vulnerabilities can often be stated as a set of constraints

$$\mathbf{RM}(\mathbf{x}) \leq \mathbf{b} \tag{1.4}$$

The vector x in (1.4) represents the holdings in a set of risky securities. The entries of the vector-valued function $\mathbf{RM}(\mathbf{x})$ represent one or more messures of risk and the vector b represents the acceptable upper limits on these measures. The set of risk mungement constraints (1.4) may be embedded in a more elaborate model that aims to optimize some kind of performance measure such as expected investment return.

Asset and Liability Management

A multi-period model that emphasizes the need to meet liabilities in each period for a finite (or possibly infinite) horizon is often more appropriate. Since liabilities and asset returns usually have random components, their optimal management requires techniques to optimize under uncertainty such as stochastic optimization. A generic asset and liabälity management model can often be formulated as a stochastic programming problem of the form

Model

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathrm{E}(U(\mathbf{x})) \\ & \mathrm{F}\mathbf{x} = \mathrm{L} \\ & \mathrm{D}\mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1.5}$$

Asset and Liability Management

- The vector $\mathbf{x}$ in (1.5) represents the investment decisions for the available assets at the dates in the planning horizon.
- The vector $\mathbf{L}$ in (1.5) represents the liabilities that the institution faces at the dates in the planning horizon
- The constraints $\mathbf{Fx} = \mathbf{L}, \mathbf{Dx} \geq 0$ represent the cash flow rules and restrictions applicable to the assets during the planning horizon.
- The term $U(\mathbf{x})$ in the objective function is some appropriate measure of utility.

For instance, it could be the value of terminal wealth at the end of the planning horizon. In general, the components F, L, D are discrete-time random processes and thus (1.5) is a multi-stage stochastic programming model with recourse. In Chapter 3 we discuss some special cases of (1.5) with no randomness.