Linear Programming: Asset–Liability Management

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overview

This chapter presents a classical application of linear programming to covering known liabilities by constructing a dedicated fixed-income portfolio. When the liabilities span multiple years, the model assumes that the only sources of risk are changes in the term structure of interest rates. We also discuss a short-term financing problem.

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Dedication

Consider the problem of funding a stream of liabilities that extends over the future. Assume the forecast of liabilities is accurate. This problem arises in certain practical situations such as the liabilities of a pension fund. It also arises in non-financial institutions planning acquisitions, expansion, or product development. A dedicated bond portfolio is a portfolio of bonds constructed today and whose cash flows offset the liabilities.

Dedication

Examples

3.1 (Bond dedication) Suppose a pension fund needs to cover some liabilities in the next six years. Cash requirements (in million \$) are:

Year	1	2	3	4	5	6
Required	100	200	800	100	800	1200

Suppose the pension fund can invest in ten government bonds with the cash flows and current prices in Table 3.1.
Find the least expensive portfolio of bonds whose cash flows will be sufficient to cover the cash requirements. Assume surplus cash can be carried from one year to the next but earn no interest.

We can formulate this problem as the following linear programming model.
Linear programming model for bond dedication

Variables:

x_j : amount of bonds j in the portfolio, for $j = 1, \ldots, 10$;

s_t : surplus cash in year t, for $t = 1, \ldots, 6$.

Table 3.1

	Year						
	1	2	3	4	5	6	Price
Bond 1	10	10	10	10	10	110	109
Bond 2	7	7	7	7	7	107	94.8
Bond 3	8	8	8	8	8	108	99.5
Bond 4	6	6	6	6	106		93.1
Bond 5	7	7	7	7	107		97.2
Bond 6	5	5	5	105			92.9
Bond 7	10	10	110				110
Bond 8	8	8	108				104
Bond 9	7	107					102
Bond 10	100						95.2

Dedication

Objective:

$$\min 109x_1 + 94.8x_2 + \cdots + 102x_9 + 95.2x_{10}$$

Constraints:

$$\begin{aligned}
&10x_1 + 7x_2 + \cdots + 7x_9 + 100x_{10} & & = 100 + s_1 \\
&10x_1 + 7x_2 + \cdots + 107x_9 & +s_1 & = 200 + s_2 \\
&\vdots & & \\
& & +s_5 & = 1200 + s_6 \\
&110x_1 + 107x_2 + 108x_3 & & \\
&x_j \geq 0, j = 1, \ldots, 10 & & \\
&s_t \geq 0, t = 1, \ldots, 6 & &
\end{aligned}$$

Dedication

Notice that we can write the equality constraints also as

$$
\begin{aligned}
&10x_1 + 7x_2 + \cdots + 7x_9 + 100x_{10} && -s_1 = 100 \\
&10x_1 + 7x_2 + \cdots + 107x_9 && +s_1 - s_2 = 200 \\
&\vdots \\
&110x_1 + 107x_2 + 108x_3 && +s_5 - s_6 = 1200 \\
&x_j \geq 0, j = 1, \ldots, 10 \\
&s_t \geq 0, t = 1, \ldots, 6
\end{aligned}
$$

or as

$$
\begin{aligned}
&10x_1 + 7x_2 + \cdots + 7x_9 + 100x_{10} && -100 = s_1 \\
&10x_1 + 7x_2 + \cdots + 107x_9 && +s_1 - 200 = s_2 \\
&\vdots \\
&110x_1 + 107x_2 + 108x_3 && +s_5 - 1200 = s_6 \\
&x_j \geq 0, j = 1, \ldots, 10 \\
&s_t \geq 0, t = 1, \ldots, 6.
\end{aligned}
$$

Dedication

In general, for a given problem with liabilities projected over m points in time over t

Date	1	2	$\cdots$	m
Required	L_1	L_2	$\cdots$	L_m

Suppose we can use n bonds with the following cash flows and prices:

Date	1	2	$\cdots$	m	Prices
Bond 1	F_{11}	F_{21}	$\cdots$	F_{m1}	p_1
$\vdots$			$\vdots$		$\vdots$
Bond j	F_{1j}	F_{2j}	$\cdots$	F_{mj}	p_j
$\vdots$			$\vdots$		
Bond n	F_{1n}	F_{2n}	$\cdots$	F_{mn}	p_n

Dedication

The linear programming formulation of the cash matching problem is as follows.
Linear programming model for bond dedication (general version)
Variables: x_j : amount of bonds j in the portfolio, for $j = 1, \ldots, n$;
s_t : surplus cash in year t, for $t = 1, \ldots, m$.
Linear programming model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{n} p_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} F_{1j} x_j - s_1 = L_1 \\
& \sum_{j=1}^{n} F_{tj} x_j + s_{t-1} - s_t = L_t, t = 2, \ldots, m \\
& x_j \geq 0, j = 1, \ldots, n \\
& s_t \geq 0, t = 1, \ldots, m
\end{array}
$$

Dedication

The problem can be written more concisely as follows:

$$\begin{array}{ll} \min & \mathbf{p}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{F}\mathbf{x} + \mathbf{R}\mathbf{s} = \mathbf{L} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{s} \geq \mathbf{0} \end{array}$$

where

$$\mathbf{R} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & \ddots & \vdots \\ F_{m1} & \cdots & F_{mn} \end{bmatrix},$$

$$\text{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, \mathbf{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}.$$

Sensitivity Analysis

As noted in Section 2.4, when a linear programming model is solved, the dual solution yields a great deal of sensitivity information, or information about what happens when data values are changed. Recall the sensitivity interpretation associated with the shadow price. Assume λ is the shadow price of a constraint:
If the right-hand side of a constraint changes by Δ, then the optimal objective value changes by $\lambda \cdot \Delta$, as long as the change of the right-hand side is within the allowable increase or decrease.

Sensitivity Analysis

This concept is particularly insightful in the bond dedication problem. In a nutshell, the sensitivity information of the linear optimization model leads to an **implied term structure** as we next explain. Recall our linear programming model for portfolio dedication:

$$
\begin{aligned}
\min \sum_{j=1}^{n} p_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} F_{1j} x_j - s_1 = L_1 \\
& \sum_{j=1}^{n} F_{tj} x_j + s_{t-1} - s_t = L_t, t = 2, \ldots, m \\
& x_j \geq 0, j = 1, \ldots, n \\
& s_t \geq 0, t = 1, \ldots, m
\end{aligned}
$$

Sensitivity Analysis

The shadow price of constraint at time t is the extra amount of money needed today to cover an extra unit of liability at time t. In other words, the shadow price λ_t gives the discount factor for time t. The current portfolio therefore implies the following term structure of interest rates:

$$r_t = \frac{1}{(\lambda_t)^{1/t}} - 1.$$

Consider again the problem of covering a stream of liabilities $(L_1, \ldots, L_m)$ due at m different dates in the future. In principle, the stream of liabilities is equivalent to a lump sum of cash today equal to its present value, obtained by discounting the future liabilities. Setting aside an amount equal to this present value seems simpler than constructing a dedicated portfolio.

A problem with this approach is that it is fully exposed to interest-rate risk. By contrast, a dedicated portfolio is not subject to interest-rate risk since it matches the liabilities at the time they occur. Immunization is an approach that reduces interest-rate risk as compared to the simple-minded present value approach, but it does not completely protect against it as dedication would. The advantage is that immunized portfolios are typically cheaper than dedicated portfolios. The idea is simple: construct a portfolio with the same present value as the stream of liabilities, and further require that this present value has the same sensitivity to changes in interest rates as the stream of liabilities.

Immunization

More precisely, suppose $r_1, \ldots, r_m$ is the term structure of risk-free interest rates. This means that the value r_t is the yield on a risk-free zero-coupon bond with maturity t. In other words, r_t is the interest rate that applies to money invested between now and time t. By discounting each of the cash flows with the appropriate discount rate, it follows that the present value (PV) of a stream of cash flows $(F_1, \ldots, F_m)$, where F_t occurs at time t, is

$$\mathrm{PV} = \frac{F_1}{1 + r_1} + \frac{F_2}{(1 + r_2)^2} + \cdots + \frac{F_m}{(1 + r_m)^m}$$

Immunization

If interest rates shift by δ, we get

$$\mathrm{PV}(\delta) = \frac{F_1}{1 + r_1 + \delta} + \frac{F_2}{(1 + r_2 + \delta)^2} + \cdots + \frac{F_m}{(1 + r_m + \delta)^m}$$

Notice that

$$\mathrm{PV}(\delta) - \mathrm{PV} \approx -\delta \left(\frac{F_1}{(1 + r_1)^2} + \frac{2F_2}{(1 + r_2)^3} + \cdots + \frac{mF_m}{(1 + r_m)^{m+1}} \right)$$

This motivates the following concept.

Immunization

Definition 3.2
The Fisher-Weil dollar duration (DD) of the stream of cash flows $(F_1, \ldots, F_m)$ is

$$\mathrm{DD} := \sum_{t=1}^{m} \frac{tF_t}{(1+r_t)^{t+1}}.$$

An immunized portfolio is a portfolio of bonds whose present value and duration match those of the stream of liabilities. In optimization terms, this corresponds to a portfolio that satisfies the following constraints:

$$\sum_{j=1}^{n} \mathrm{PV}_j x_j = \mathrm{PV}_L$$

$$\sum_{j=1}^{n} \mathrm{DD}_j x_j = \mathrm{DD}_L$$

Immunization

A closer look at the difference between PV and $\mathrm{PV}(\delta)$ suggests that we can get an even better matching of sensitivity to changes in the term structure by looking at second-order terms:

$$\mathsf{PV}(\delta) - \mathrm{PV} \approx -\delta \sum_{t=1}^{m} \frac{tF_t}{(1+r_t)^{t+1}} + \frac{1}{2}\delta^2 \sum_{t=1}^{m} \frac{t(t+1)F_t}{(1+r_t)^{t+2}}$$

This leads to the so-called Fisher-Weil dollar convexity (DC) of $(F_1, \ldots, F_m)$:

$$\mathrm{DC} := \sum_{t=1}^{m} \frac{t(t+1)F_t}{(1+r_t)^{t+2}}$$

as well as the Fisher-Weil convexity (C):

$$\mathrm{C} := \frac{1}{\mathrm{PV}} \sum_{t=1}^{m} \frac{t(t+1)F_t}{(1+r_t)^{t+2}}$$

Immunization

A portfolio can therefore be further immunized by matching present value, dollar duration, and dollar convexity:

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{PV}_j x_j &= \mathrm{PV}_L \\ \sum_{j=1}^{n} \mathrm{DD}_j x_j &= \mathrm{DD}_L \\ \sum_{j=1}^{n} \mathrm{DC}_j x_j &\geq \mathrm{DC}_L \end{aligned} \tag{3.1}$$

Here the subindices $j = 1, \ldots, n$ and L refer to the bonds and liabilities respectively. Note that since having net positive convexity is favorable, the last constraint is an inequality constraint.

The immunization constraints (3.1) are generally less stringent than the bond dedication constraints, namely

$$\begin{aligned}
&\sum_{j=1}^{n} F_{1j}x_j - s_1 = L_1 \\
&\sum_{j=1}^{n} F_{tj}x_j + s_{t-1} - s_t = L_t, t = 2, \ldots, m \\
&x_j \geq 0, j = 1, \ldots, n \\
&s_t \geq 0, t = 1, \ldots, m
\end{aligned} \tag{3.2}$$

Indeed, if the surplus variables $s_t, t = 1, \ldots, m$, are all zero in (3.2), then some straightforward algebra shows that any $x_1, \ldots, x_n$ satisfying (3.2) also satisfies (3.1).

Immunization

The previous discussion assumes that interest is compounded at discrete time intervals, e.g., annually or semiannually. In some practical circumstances cash flows may occur at irregular times. In those cases it could be more convenient to assume that interest is continuously compounded. Suppose r_t is the continuously compounded spot rate for a risk-free zero-coupon bond with maturity t. Then the present value of a stream of cash flows $(F_1, \ldots, F_m)$ is

$$\mathrm{PV} = \sum_{t=1}^{m} F_t e^{-t \cdot r_t}$$

Immunization

Consequently its dollar duration is

$$\mathrm{DD} = \sum_{t=1}^{m} tF_t e^{-t \cdot r_t}$$

and its dollar convexity is

$$\mathrm{DC} = \sum_{t=1}^{m} t^2 F_t e^{-t \cdot r_t}.$$

Immunization

A nice feature of continuous compounding is that the formulas for an irregular stream of cash flows $(F_{t_1}, \ldots, F_{t_m})$ are very similar:

$$\mathrm{PV} = \sum_{i=1}^{m} F_{t_i} e^{-t_i \cdot r_{t_i}}$$

$$\mathrm{DD} = \sum_{i=1}^{m} t_i F_{t_i} e^{-t_i \cdot r_{t_i}}$$

$$\mathrm{DC} = \sum_{i=1}^{m} t_i^2 F_{t_i} e^{-t_i \cdot r_{t_i}}$$

Immunization

The kind of immunization via duration and convexity enforced by the constraints (3.1) provides hedging against parallel shocks in the term structure.

This implicitly assumes a one-factor interest risk model. There are enhancements based on a multi-factor interest risk model. Two popular ones are the **key-rate model** and the **shift–twist–butterfly model** as discussed in Tuckman (2002). The logic of immunization naturally extends to a multi-factor interest risk model. In such a context an immunized portfolio should be hedged against changes in each of the risk factors.

Some Practical Details about Bonds

There are certain details about the way bonds are quoted and traded in actual exchanges. The discussion below applies only to plain vanilla treasury bonds.

Principal Value, Coupon Payments, Clean and Dirty Prices

The **principal value, or par, or principal** of a bond is the amount that the issuer agrees to repay the bondholder.

The **term to maturity** of a bond is the time remaining until principal payment.

The **maturity date** of a bond is that date when the issuer will pay the principal.

The **coupon rate or nominal rate** of a bond is the annual interest that the issuer pays the bondholder. Treasury bonds pay their coupons semiannually. For example, a bond with an 8% coupon rate and a principal of $\$1,000$ will pay a $\$40$ installment to the holder every six months. At the maturity date, it will pay the $\$40$ installment plus the $\$1,000$ principal.

Some Practical Details about Bonds

Principal Value, Coupon Payments, Clean and Dirty Prices

When an investor purchases a bond between coupon payments, the investor must compensate the seller of the bond for the coupon interest earned since the last coupon payment. This is called the accrued interest and is computed based on the proportion of time since the last coupon payment. The convention for United States treasuries is not to include the accrued interest in the price quote. This price is called the clean price or simply the price. It is customary to present the price quote as a percentage of the par value of the bond. The clean price plus the accrued interest is called the **dirty price or full price**.

Principal Value, Coupon Payments, Clean and Dirty Prices

For example, suppose that on February 15, 2051 investor B buys a treasury bond with $\$10,000$ face value, 5.5% coupon rate that matures on January 31, 2053. In this case the coupon payment is $\$275 = 2.75\%$ of $\$10,000$ and the accrued interest is

$$\frac{15}{181} \cdot 275 = 22.79$$

Suppose the price quote on February 15, 2051 is 101.145 . Then the full price of the bond, that is, the price paid by the buyer to the seller, is $\$10,114.5 + \$22.79 = \$10,137.29$.

Some Practical Details about Bonds

Yield Curve and Term Structure

Recall that the yield of a bond is the interest rate that makes the discounted value of the cash flows match the current price of the bond. By convention, the yield is quoted on an annual basis. The **treasury yield curve** is the curve of yields for **on-the-run** (most recently auctioned) treasuries. It should be noted that the yield curve is not the same as the term structure of interest rates. This is the case because treasuries with maturity greater than one year are not zero-coupon bonds. Indeed, the term structure of interest rates is actually a theoretical construct that must be estimated from actual bonds.

Some Practical Details about Bonds

Yield Curve and Term Structure

There are various ways of estimating the term structure. A quick and dirty (perhaps too dirty) approach is to ignore the difference described above and to use the yield curve as a proxy for the term structure of interest rates.

A second approach is to use the following **bootstrapping approach**: Use several coupon-bearing bonds with various maturities. Determine the spot rate implied by the bond with the shortest maturity. Use that knowledge to compute the spot rate implied by the bond with the next shortest maturity and so on.

Yield Curve and Term Structure

For example, suppose we have a 0.5 -year 5.25% bill, a 1-year 5.75% note, and a 1.5-year 6% note. For simplicity assume they all are trading at par. Let z_1, z_2, z_3 denote the one-half annualized 0.5 -year, 1 -year, and 1.5 -year spot rates.

Using the 0.5 -year bond, we readily get

$$z_1 = 0.0525 \cdot 0.5 = 0.02625$$

Using \$100 as par, the cash flows for the 1-year bond are

$$\begin{aligned} &\text{0.5-year: } 0.0575 \cdot 100 \cdot 0.5 = 2.875 \\ &\text{1-year: } 0.0575 \cdot 100 \cdot 0.5 + 100 = 102.875. \end{aligned}$$

Some Practical Details about Bonds

Yield Curve and Term Structure

Now compute its present value using the spot rates z_1, z_2 and equate that to its current price:

$$100 = \frac{2.875}{1+z_1} + \frac{102.875}{(1+z_2)^2}$$

Because we already know z_1, we can solve for z_2 and obtain

$$z_2 = 0.028786$$

Repeat with the 1.5-year bond: Using \$100 as par, the cash flows for the 1.5 -year bond are
0.5 -year: $0.06 \cdot 100 \cdot 0.5 = 3$
1-year: $0.06 \cdot 100 \cdot 0.5 = 3$
1.5 -year: $0.06 \cdot 100 \cdot 0.5 + 100 = 103.$

Some Practical Details about Bonds

Yield Curve and Term Structure

Now compute its present value using the spot rates z_1, z_2, z_3 and equate that to its current price:

$$100 = \frac{3}{1+z_1} + \frac{3}{(1+z_2)^2} + \frac{103}{(1+z_3)^3}.$$

Because we already know z_1, z_2, we can solve for z_3 and obtain

$$z_3 = 0.030063097$$

Thus the annualized spot rates are

$$r_{0.5} = 0.0525, r_1 = 0.057572, r_2 = 0.06012.$$

Some Practical Details about Bonds

Yet a third, and much more elaborate, approach to estimating the term structure is to take into consideration all bonds with similar characteristics available in the market and perform an elaborate regression model. This approach requires advanced statistical techniques and is beyond the scope of this book. For a related discussion see Campbell et al. (1997) and Heath et al. (1992)

It should also be noted that the previous two estimation approaches only give spot rates at specific points in time. The spot rates at other times can be obtained by interpolation. The simplest type of interpolation is piecewise linear.

Other Cash Flow Problems

The dedication model discussed in Section 3.1 belongs to the broader class of cash flow problems. A firm faces a stream of both positive (inflows) and negative (outflows) flows of cash. The negative flows are considered liabilities that must be met when they occur. To meet the liabilities, the firm can purchase a variety of instruments each with a different cash flow pattern.

The following short-term financing problem is of this kind. Corporations routinely face the problem of financing short-term cash commitments. Linear programming can help in figuring out an optimal combination of financial instruments to meet these commitments. For illustration, consider the following problem. For the sake of exposition, we keep the example small.

Examples

3.3 (Short-term financing) A company has the following short-term financing problem (net cash flow requirements are given in \$1000 s).

Month	J	F	M	A	M	J
Net cash flow	-150	-100	200	-200	50	300

The company has the following sources of funds:

- A line of credit of up to $\$100,000$ at an interest rate of 1% per month.
- It can issue 90 -day commercial paper bearing a total interest of 2% for the 3 -month period.
- Each month excess funds can be invested at an interest rate of 0.3% per month.

Examples

3.3 (Short-term financing)
Linear programming model for short-term financing problem

Variables:

x_j : amount drawn from the line of credit in month j, for $j = 1, \ldots, 5$
y_j : amount of commercial paper issued in month j, for $j = 1, \ldots, 3$
z_j : excess funds in month j, for $j = 1, \ldots, 6$.

Objective:

$$\max z_6$$

Examples

3.3 (Short-term financing)

Constraints:

Cash balance constraints in each month and bounds on x_j, y_j and z_j :

$$
\begin{aligned}
x_1 + y_1 \quad - z_1 &= 150 \\
x_2 + y_2 - 1.01x_1 + 1.003z_1 - z_2 &= 100 \\
x_3 + y_3 - 1.01x_2 + 1.003z_2 - z_3 &= -200 \\
x_4 - 1.02y_1 - 1.01x_3 + 1.003z_3 - z_4 &= 200 \\
x_5 - 1.02y_2 - 1.01x_4 + 1.003z_4 - z_5 &= -50 \\
- 1.02y_3 - 1.01x_5 + 1.003z_5 - z_6 &= -300 \\
x_j \leq 100 \text{ for } j &= 1, \ldots, 5 \\
x_j \geq \quad 0 \text{ for } j &= 1, \ldots, 5 \\
y_j \geq \quad 0 \text{ for } j &= 1, \ldots, 3 \\
z_j \geq \quad 0 \text{ for } j &= 1, \ldots, 5.
\end{aligned}
$$

Other Cash Flow Problems

Examples

Solving this linear program using either Excel Solver or MATLAB CVX, we obtain the following optimal solution:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 52 \end{bmatrix}, \quad \mathbf{y}^* = \begin{bmatrix} 150 \\ 100 \\ 151.944 \end{bmatrix}, \quad \mathbf{z}^* = \begin{bmatrix} 0 \\ 0 \\ 351.944 \\ 0 \\ 0 \\ 92.497 \end{bmatrix}$$

Thus the company can attain an optimal wealth of $\$92,497$ in June. To achieve this, the company will issue $\$150,000$ in commercial paper in January, $\$100,000$ in February and $\$151,944$ in March. In addition, it will draw $\$52,000$ from its line of credit in May. Excess cash of $\$351,944$ in March will be invested for one month.

Other Cash Flow Problems

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
I8	credit J Amount	0	-0.003214	0	0.00321386	1E+30
I9	credit F Amount	0	-3.89E-16	0	3.8858E-16	1E+30
I10	credit M Amount	0	-0.007119	0	0.00711864	1E+30
I11	credit A Amount	0	-0.003151	0	0.00315085	1E+30
I12	credit M Amount	52	0	0	0.00311965	3.8096E-16
I13	commercial J Amount	150	0	0	0.00399754	0.00321386
I14	commercial F Amount	100	0	0	0.00318204	3.8858E-16
I15	commercial M Amount	151.9441675	0	0	3.8473E-16	0.0031603
B18	Surplus January	0	-0.003998	0	0.00399754	1E+30
C18	Surplus February	0	-0.00714	0	0.00714	1E+30
D18	Surplus March	351.9441675	0	0	0.00393091	0.0031603
E18	Surplus April	0	-0.003919	0	0.00391915	1E+30
F18	Surplus May	0	-0.007	0	0.007	1E+30
G18	Surplus June	92.49694915	0	1	1E+30	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B19	Total January	150	-1.037288	150	89.1718954	149.411765
C19	Total February	100	-1.0302	100	47.0588235	50.9803922
D19	Total March	-200	-1.02	-200	90.6832835	151.944167
E19	Total April	200	-1.016949	200	90.9553333	152.4
F19	Total May	-50	-1.01	-50	48	52
G19	Total June	-300	-1	-300	92.4969492	1E+30

Figure: 3.1 Sensitivity report for short-term financing model

Figure 3.1 displays the Excel Solver sensitivity report for this model. The key columns for sensitivity analysis are the “Reduced Cost” and “Shadow Price” columns.

Recall that the shadow price u of a constraint C has the following interpretation:

If the right-hand side of the constraint C changes by an amount Δ, the optimal objective value changes by $u \cdot \Delta$ as long as Δ is within a certain range.

Other Cash Flow Problems

- For example, assume that net cash flow in January were -200 (instead of -150). By how much would the wealth of the company decrease at the end of June? The answer is in the shadow price of the January constraint, $u = -1.0373$. The right-hand side of the January constraint would go from 150 to 200 , an increase of $\Delta = 50$, which is within the allowable increase (89.17). So the wealth of the company in June would decrease by $1.0373 \cdot 50,000 = \$51,865$.
- Now assume that net cash flow in March were 250 (instead of 200). By how much would the wealth of the company increase at the end of June? Again, the change $\Delta = -50$ is within the allowable decrease (151.944), so we can use the shadow price $u = -1.02$ to calculate the change in objective value. The increase is $(-1.02) \cdot (-50) = \$51,000$.

Other Cash Flow Problems

- Assume that the negative net cash flow in January is due in part to the purchase of a machine worth $\$100,000$. The vendor allows the payment to be made in June at an interest rate of 3% for the 5 -month period. Would the wealth of the company increase or decrease by using this option? What if the interest rate for the 5 -month period were 4% ? The shadow price of the January constraint is -1.0373 . This means that reducing cash requirements in January by \$1 increases the wealth in June by \$1.0373. In other words, the break-even interest rate for the 5 -month period is 3.73%. So, if the vendor charges 3%, we should accept, but if he charges 4% we should not. Note that the analysis is valid since the amount $\Delta = -100$ is within the allowable decrease.

Next, let us consider the reduced costs. Recall that these are the shadow prices of the upper and lower bounds placed directly on the variables. The reduced cost of a variable is non-zero only when the variable is equal to one of its bounds. Assume x is equal to its lower bound b and its reduced cost is c. There are two useful interpretations of the reduced cost c.

- First, if the value of x is set to a value $b + \Delta$ for $\Delta > 0$ instead of its optimal value b then the objective value is changed by $c \cdot \Delta$. For example, what would be the effect of financing part of the January cash needs through the line of credit? The answer is in the reduced cost of the first variable. Because this reduced cost -0.0032 is strictly negative, the objective function would decrease. Specifically, each dollar financed through the line of credit in January would result in a decrease of $3.2 in the wealth of the company in June.

Other Cash Flow Problems

- The second interpretation of c is that its magnitude $|c|$ is the minimum amount by which the objective coefficient of x must be changed in order for the variable x to move away from its bound in an optimal solution. For example, consider the first variable again. Its value is zero in the current optimal solution, with objective function z_6. However, if we changed the objective to $z_6 + 0.0032x_1$, it would now be optimal to use the line of credit in January. In other words, the reduced cost on x_1 can be viewed as the minimum rebate that the bank would have to offer (payable in June) to make it attractive to use the line of credit in January.