Linear Programming: Arbitrage and Asset Pricing

Sirong Luo

Faculty of Statistics and Data Science
Shanghai University of Finance and Economics

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Arbitrage Detection in the Foreign Exchange Market

The foreign exchange market includes the trading of currencies. It is one of the markets with largest trading volume. Given two currencies at any particular time, say the US dollar and the euro, there are two exchange rates between them: one dollar will buy r_1 euros, and one euro will buy r_2 dollars. It is evident that an arbitrage opportunity would arise if $r_1 r_2 > 1$ since one could simultaneously convert 1 dollar into r_1 euros and the r_1 euros into $r_1 r_2 > 1$ dollars. These two transactions would net $r_1 r_2 - 1$ dollars without any risk.

Arbitrage Detection in the Foreign Exchange Market

An interesting related question is: Can one detect a similar type of arbitrage opportunity involving more than two currencies? In particular, consider the following hypothetical exchange rates among the currencies USD(US Dollars), EUR (Euros), GBP (British Pounds), AUD (Australian Dollars), and JPY(Japanese Yen).

	USD	EUR	GBP	AUD	JPY
USD	1	0.639	0.537	1.0835	98.89
EUR	1.564	1	0.843	1.6958	154.773
GBP	1.856	1.186	1	2.014	184.122
AUD	0.9223	0.589	0.496	1	91.263
JPY	0.01011	0.00645	0.00543	0.01095	1

Arbitrage Detection in the Foreign Exchange Market

We next show how to answer these questions using linear programming. For convenience, use $i = 1, \ldots, 5$ to index the above five currencies USD, EUR, GBP, AUD, and JPY in that order. We let a_{ij} denote the exchange rate from currency i to currency j. For instance $a_{34} = 2.014$ and $a_{25} = 154.773$. To model a set of transactions with potential for arbitrage, consider the following decision variables:

- x_{ij} : amount of currency i converted to currency j.
- y_k : net amount of currency k after all transactions.

These variables are related via the following constraints:

$$y_k = \sum_{i=1}^{5} a_{ik} x_{ik} - \sum_{j=1}^{5} x_{kj}, k = 1, \ldots, 5$$

Arbitrage Detection in the Foreign Exchange Market

An arbitrage would exist if there is a set of transactions so that after all transactions the net amount for each currency is non-negative and at least one of them is strictly positive. To find such a set of transactions we could solve the following linear programming problem:

$$
\begin{array}{ll}
\max & y_1 \\
\text{s.t.} & y_k = \sum_{i=1}^{5} a_{ik} x_{ik} - \sum_{j=1}^{5} x_{kj}, \; k = 1, \ldots, 5 \\
& x_{ij} \geq 0 \\
& y_k \geq 0
\end{array}
$$

Arbitrage Detection in the Foreign Exchange Market

However, if there is indeed an arbitrage opportunity, then the above problem would be unbounded. We can easily amend the above model so that the arbitrage can be revealed by introducing a bound on the objective function:

$$
\begin{array}{ll}
\max & y_1 \\
\text{s.t.} & y_k = \sum_{i=1}^{5} a_{ik} x_{ik} - \sum_{j=1}^{5} x_{kj}, \; k = 1, \ldots, 5 \\
& y_1 \leq 1 \\
& x_{ij} \geq 0 \\
& y_k \geq 0.
\end{array}
$$

Solving this linear programming model, we find that indeed there are arbitrage opportunities. However, to obtain $1 in arbitrage, we have to exchange about 1669.172 US dollars into 1066.601 euros, then convert these euros into 899.1446 pounds, then convert these pounds into 1810.877 Australian dollars, and finally change these Australian dollars into 1670.172 US dollars. The arbitrage opportunity is so tight that, depending on the numerical precision used, a linear programming solver may not find it. Furthermore, even if a solver does find it, the tightness of the arbitrage may render it impractical when accounting for market frictions such as transaction costs.

The Fundamental Theorem of Asset Pricing

One of the most widely studied problems in financial mathematics is the pricing of contingent claims. These are securities whose price depends on the value of another underlying security. Under the assumption of no arbitrage, the price of such a contingent claim should match the price of a portfolio that replicates the payoff of the contingent claim. This basic principle underlies the powerful option pricing machinery dating back to the pioneering work of Merton (1973) and Black and Scholes (1973). The absence of arbitrage and the replication argument can be cleverly stated in terms of a so-called **risk-neutral probability measure**. The latter concept can be equivalently stated in terms of **a stochastic discount factor or a positive linear pricing rule**.

The Fundamental Theorem of Asset Pricing

We next use linear programming duality to give a formal derivation of the equivalence between the absence of arbitrage and the existence of a risk-neutral probability measure for the special case of a simple economy in a single-period framework. Assume the economy contains m assets. Let $\mathbf{S}_0 := \begin{bmatrix} S_0^1 & \cdots & S_0^m \end{bmatrix}^\top$ denote the vector of prices per share of the m assets at time 0 (beginning of the period). Assume there are n possible states $\Omega = \{\omega_1, \ldots, \omega_n\}$ at time 1 (end of the period). Let $\mathbf{S}_1(\omega_j) = \begin{bmatrix} S_1^1(\omega_j) & \cdots & S_1^m(\omega_j) \end{bmatrix}^\top$ denote the vector of prices per share of the m assets at time 1 in state ω_j.

The Fundamental Theorem of Asset Pricing

An arbitrage opportunity in this economy is an opportunity to make money without any cost and without any risk. Mathematically, an arbitrage opportunity is a portfolio of the m assets that has non-positive cost, yields non-negative payoffs in all future states, and in addition either has strictly negative cost or generates a strictly positive payoff in some future state. In other words, an arbitrage portfolio is a set of holdings $y_1, \ldots, y_m$ in the m assets such that

$$\mathbf{S}_0^\top \mathbf{y} \leq 0, \quad \mathbf{S}_1 (\omega_j)^\top \mathbf{y} \geq 0, \quad j = 1, \ldots, n$$

and such that at least one of these inequalities is strict. A positive linear pricing rule is a set of positive numbers $x_1, \ldots, x_n$ such that

$$\mathbf{S}_0 = \sum_{j=1}^{n} \mathbf{S}_1 (\omega_j) x_j, \quad i = 1, \ldots, m$$

The Fundamental Theorem of Asset Pricing

Proposition 4.1 In the above single-period economy with m assets and n future states there is no arbitrage if and only if there exists a positive linear pricing rule.

Proof Let $\mathbf{S} := \begin{bmatrix} \mathbf{S}_1(\omega_1) & \cdots & \mathbf{S}_1(\omega_n) \end{bmatrix}$. An arbitrage portfolio is precisely a solution to the following system of inequalities:

$$[\mathbf{S} - \mathbf{S}_0]^\top \mathbf{y} > \mathbf{0} \tag{4.1}$$

Similarly, a positive linear pricing rule is precisely a solution to the following system of inequalities:

$$\begin{aligned} \mathbf{S}\mathbf{x} &= \mathbf{S}_0 \\ \mathbf{x} &> \mathbf{0} \end{aligned} \tag{4.2}$$

The Fundamental Theorem of Asset Pricing

Observe that (4.2) has a solution if and only if the following system of inequalities has a solution:

$$\begin{aligned} \begin{bmatrix} \mathbf{S} & -\mathbf{S}_0 \end{bmatrix} \mathbf{u} &= \mathbf{0} \\ \mathbf{u} &> \mathbf{0} \end{aligned} \tag{4.3}$$

Hence it suffices to show that (4.1) does not have a solution if and only if (4.3) has a solution. This readily follows from Theorem 2.5(c).

The Fundamental Theorem of Asset Pricing

The existence of a positive linear pricing rule can be equivalently stated in terms of a stochastic discount factor or in terms of a risk-neutral measure. In both of these interpretations the set of future states $\Omega = \{\omega_1, \ldots, \omega_n\}$ is seen as a probability space. Assume Ω is endowed with a probability measure $\mathbb{P}$. Then the future payoff of each asset i can be seen as a random variable $S_i : \Omega \to \mathbb{R}$. A stochastic discount factor is a random variable $D : \Omega \to \mathbb{R}$ such that

$$\mathbf{S}_0 = \mathbb{E}\left(D\mathbf{S}_1\right) = \sum_{j=1}^{n} D\left(\omega_j\right) \mathbf{S}_1\left(\omega_j\right) \mathbb{P}\left(\omega_j\right), \quad i = 1, \ldots, m$$

For convenience, assume there is a risk-free asset in the above economy; that is, an asset i such that $S_0^i = 1$ and $S_1^i(\omega_j) = 1 + r$ for $j = 1, \ldots, n$. A risk-neutral probability measure is a probability measure $\mathbb{Q}$ in the space $\Omega = \{\omega_1, \ldots, \omega_n\}$ such that

$$\mathbf{S}_0 = \frac{1}{1+r}\widetilde{\mathbb{E}}(\mathbf{S}_1) = \frac{1}{1+r}\sum_{j=1}^{n}\mathbf{S}_1(\omega_j)\,\mathbb{Q}(\omega_j).$$

Here $\widetilde{\mathbb{E}}$ indicates that the expectation is taken with respect to the risk-neutral probability measure $\mathbb{Q}$, as opposed to the original probability measure $\mathbb{P}$.

We can now formally state the fundamental theorem of asset pricing.

The Fundamental Theorem of Asset Pricing

Theorem 4.2

Fundamental theorem of asset pricing Consider the above single-period economy with n future states and m assets, one of which is risk-free. The following conditions are equivalent:

- There are no arbitrage opportunities.
- There exists a positive linear pricing rule.
- There exists a positive stochastic discount factor.
- There exists a risk-neutral probability measure.

Proposition 4.1, which gives the equivalence between (i) and (ii), provides the crux of the proof of Theorem 4.2. The proofs of the other equivalences are a straightforward exercise.

One-Period Binomial Pricing Model

This section illustrates the pricing of a contingent claim on an underlying risky security in a simple one-period binomial model. This model provides the building block for the powerful and widely used multi-period binomial pricing model that we will discuss in Chapter 15. Consider a single-period economy with a risk-free asset and a risky asset. Let r denote the risk-free rate and S_0 denote the price per share of the risky asset at time 0 . Assume there are two possible future states $\Omega = \{H, T\}$ at time 1 . Assume the price per share of the risky asset at time 1 is $S_1(H) = u \cdot S_0$ in state H and $S_1(T) = d \cdot S_0$ in state T for some "up" and "down" factors $u > d > 0$:

$$\begin{tikzcd}[row sep=small] & S_1(H) = uS_0 \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_1(T) = dS_0 \end{tikzcd}$$

In this economy there is no arbitrage if and only if $u > 1 + r > d$ and in this case the risk-neutral probability measure should satisfy

$$
\begin{aligned}
S_0 &= \frac{1}{1+r}\left(\mathbb{Q}(H)S_1(H) + \mathbb{Q}(T)S_2(T)\right) = \frac{S_0}{1+r}(\mathbb{Q}(H)u + \mathbb{Q}(T)d) \\
1 &= \mathbb{Q}(H) + \mathbb{Q}(T)
\end{aligned}
$$

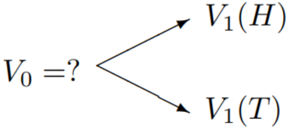
Therefore,

$$
\mathbb{Q}(H) = \frac{1+r-d}{u-d}, \quad \mathbb{Q}(T) = \frac{u-1-r}{u-d}. \tag{4.4}
$$

It is customary to write $\tilde{p} := \mathbb{Q}(H)$ and $\tilde{q} := 1 - \tilde{p} = \mathbb{Q}(T)$ as shorthand for the risk-neutral probabilities and $p = \mathbb{P}(H)$ and $q = 1 - p = \mathbb{P}(T)$ for the actual probabilities:

One-Period Binomial Pricing Model

Consider the problem of pricing a contingent claim on the risky asset with the following payoff structure:



For example, the contingent claim could be a European call option - that is, a contract with the following conditions. At time 1, the holder of the option has the right, but not the obligation, to purchase a share of the risky asset, known as the underlying security, for a prescribed amount, known as the strike price. Thus the payoff of a European call option with strike K is $V_1 = (S_1 - K)^+ := \max\{S_1 - K, 0\}$.

One-Period Binomial Pricing Model

The payoff structure of this option in our one-period binomial model is as follows:

$$\begin{tikzcd} & V_1(H) = (uS_0 - K)^+ \\ V_0 = ? \arrow[ur] \arrow[dr] & \\ & V_1(T) = (dS_0 - K)^+ \end{tikzcd}$$

A European put option is a similar contract, except that it confers the right to sell the underlying security for a prescribed strike price.
The fundamental theorem of asset pricing implies that the fair price V_0 of a general contingent claim with payoffs $V_1(H)$ and $V_1(T)$ is

$$V_0 = \frac{1}{1+r}\left(\tilde{p}V_1(H) + \tilde{q}V_1(T)\right).$$

One-Period Binomial Pricing Model

Furthermore, the binomial pricing model yields the following delta-hedging formula to construct a portfolio of the underlying risky asset and the risk-free asset that replicates the payoff of the contingent claim. At time 0 construct a portfolio with Δ shares of the underlying risky asset and B shares of the risk-free asset where

$$\Delta := \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{V_1(H) - V_1(T)}{S_0(u-d)}, \quad B := \frac{uV_1(T) - dV_1(H)}{(1+r)(u-d)}.$$

A straightforward verification shows that this portfolio replicates the payoff of the contingent claim. That is, the payoff of the portfolio (Δ, B) is as follows:

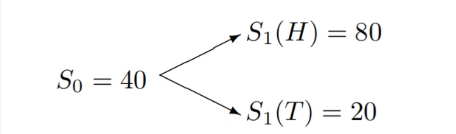
$$
\begin{tikzcd}[column sep=large]
 & \Delta S_1(H) + (1+r)\,B = V_1(H) \\
\Delta S_0 + B \arrow[ur] \arrow[dr] & \\
 & \Delta S_1(T) + (1+r)\,B = V_1(T)
\end{tikzcd}
$$

Thus the value of this replicating portfolio at time 0 must be V_0 to rule out arbitrage. Indeed, the value of the replicating portfolio at time 0 is

$$
\begin{aligned}
\Delta S_0 + B &= \frac{(1+r)\,(V_1(H) - V_1(T)) + uV_1(T) - dV_1(H)}{(1+r)(u-d)} \\
&= \frac{1}{1+r}\,(\tilde{p}V_1(H) + \tilde{q}V_1(T)) = V_0.
\end{aligned}
$$

Example

Example 4.3 Suppose stock XYZ has share price $S_0 = 40$ today. Suppose the share price of stock XYZ a month from today will either double or halve with equal probabilities:



Assume also that the one-month risk-free rate is zero. Consider a European call option to buy one share of XYZ stock for \$50 a month from today. What is the fair price of this option?

One-Period Binomial Pricing Model

In Example 4.3 we have $u = 2, d = \frac{1}{2}$ and $r = 0$. Thus the risk-neutral probabilities are $\tilde{p} = \frac{1}{3}$ and $\tilde{q} = \frac{2}{3}$. Next, observe that a month from now the call option with strike price \$50 will be worth \$30 = \$80 − \$50 in the H state and it will be worthless in the T state. Thus the fair price of the option is the price of the following contract:

$$? \begin{array}{l} \nearrow (80-50)^+ = 30 \\ \\ \searrow (20-50)^+ = 0 \end{array}$$

The fundamental theorem of asset pricing implies that the fair price of this contract is

$$30 \cdot \tilde{p} + 0 \cdot \tilde{q} = 30 \cdot \frac{1}{3} = 10$$

One-Period Binomial Pricing Model

Furthermore, from the delta-hedging formula it follows that a replicating portfolio can be constructed by buying $\frac{1}{2}$ share of stock XYZ and borrowing 10 shares of the risk-free asset. Observe that the value of this replicating portfolio at time 0 is

$$\frac{1}{2} \cdot 40 - 10 = 10$$

Using the risk-neutral probability measure we can also price other derivative securities on the XYZ stock. For example, consider a European put option on the XYZ stock with strike price \$60 and with the same expiration date:

$$? \begin{cases} \nearrow (60 - 80)^+ = 0 \\ \searrow (60 - 20)^+ = 40 \end{cases}$$

It readily follows that the fair price of this option is

$$0 \cdot \tilde{p} + 40 \cdot \tilde{q} = 40 \cdot \frac{2}{3} = \frac{80}{3}$$

Observe that in the one-period binomial pricing model the risk-neutral probability is unique and the payoff of any contingent claim can be replicated via delta-hedging. In general, uniqueness of the risk-neutral probability corresponds to completeness of the market. The latter concept means that the payoff of any contract can be replicated with a portfolio of the existing underlying assets in the economy as detailed in Exercise 4.6.

Static Arbitrage Bounds

The no-arbitrage approach discussed in Section 4.2 has the drawback that it assumes only a finite number of possible future states. In this section, we do not make this assumption. Instead, **we assume that there is a finite set of derivative securities written on the same underlying asset and with the same maturity.** We show how the no-arbitrage approach can be used to obtain so-called **static arbitrage bounds** on the price of a new derivative security implied by the prices of the other derivative securities. As in Section 4.2, the gist of this approach is to use linear programming to detect arbitrage opportunities in a single-period economy. This discussion is based on Herzel (2005).

Static Arbitrage Bounds

Consider an underlying security with a (random) price S_T at a future time T. Consider n derivative securities written on this security that mature at time T, and have piecewise linear payoff functions $\Psi_i(S_T)$, each with a single breakpoint K_i, for $i = 1, \ldots, n$. The obvious motivation is the collection of calls and puts written on the underlying security with strike prices $K_i, i = 1, \ldots, n$. More precisely, if the i th derivative security were a European call with strike price K_i, we would have $\Psi_i(S_T) = (S_T - K_i)^+$. If it were a European put with strike price K_i, we would have $\Psi_i(S_T) = (K_i - S_T)^+$.

Static Arbitrage Bounds

We shall assume without loss of generality that the K_i s are in increasing order. Also, we let p_i denote the current price of the i th derivative security. Consider a portfolio $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ of the derivative securities 1 to n and let $\Psi^{\mathbf{x}}(S_T)$ denote the payoff function of the portfolio:

$$\Psi^{\mathbf{x}}(S_T) = \sum_{i=1}^{n} \Psi_i(S_T)\, x_i$$

The cost of the portfolio x is given by

$$\sum_{i=1}^{n} p_i x_i \tag{4.5}$$

Static Arbitrage Bounds

To determine whether there exists an arbitrage opportunity in the above set of n derivative securities, we consider the following question: Is it possible to construct a portfolio of the derivative securities $1, \ldots, n$ with negative cost and whose payoff function $\Psi^{\mathsf{x}}(S_T)$ at time T is non-negative for all $S_T \in [0, \infty)$? Since non-negativity of $\Psi^{\mathsf{x}}(S_T)$ corresponds to "no future obligations" such a portfolio would be an arbitrage opportunity.

Since all $\Psi_i(S_T)\mathrm{s}$ are piecewise linear, so is $\Psi^{\mathrm{x}}(S_T)$ with breakpoints in $K_1, \ldots, K_n$. Note that a piecewise linear function is non-negative over $[0, \infty)$ if and only if it is non-negative at 0 and all the breakpoints, and if the slope of the function is non-negative to the right of the largest breakpoints.

Static Arbitrage Bounds

In other words, $\Psi^{\mathrm{x}}(S_T)$ is non-negative for all $S_T \geq 0$ if and only if the following three conditions hold:

- $\Psi^{\mathbf{x}}(0) \geq 0$
- $\Psi^{\mathbf{x}}(K_j) \geq 0, j = 1, \ldots, n$
- $\left[(\Psi^{\mathrm{x}})'_{+}(K_n)\right] \geq 0$

These three conditions can be written as the following system of linear inequalities:

$$\sum_{i=1}^{n} \Psi_i(0) x_i \geq 0$$

$$\sum_{i=1}^{n} \Psi_i(K_j) x_i \geq 0, \quad j = 1, \ldots, n \tag{4.6}$$

$$\sum_{i=1}^{n} (\Psi_i(K_n + 1) - \Psi_i(K_n)) x_i \geq 0$$

Static Arbitrage Bounds

Since all $\Psi_i(S_T)$ s are piecewise linear, the quantity $\Psi_i(K_n + 1) - \Psi_i(K_n)$ gives the right derivative of $\Psi_i(S_T)$ at K_n and the expression in the last constraint is the right derivative of $\Psi^{\mathbf{x}}(S_T)$ at K_n. The system of linear inequalities (4.6) can be more succinctly written as

$$\mathbf{Kx} \geq \mathbf{0}$$

for

$$\mathbf{K} := \begin{bmatrix} \Psi_1(0) & \cdots & \Psi_n(0) \\ \Psi_1(K_1) & \cdots & \Psi_n(K_1) \\ \vdots & & \vdots \\ \Psi_1(K_n) & \cdots & \Psi_n(K_n) \\ \Psi_1(K_n+1) - \Psi_1(K_n) & \cdots & \Psi_n(K_n+1) - \Psi(K_n) \end{bmatrix}$$

Static Arbitrage Bounds

It thus follows that the above type of arbitrage opportunity exists if and only if the following problem has a solution:

$$\mathbf{K}\mathbf{x} \geq \mathbf{0}, \quad \mathbf{p}^\top \mathbf{x} < 0.$$

Next, we focus on the special case where the derivative securities under consideration are European call options with strikes K_i for $i = 1, \ldots, n$. In this case $\Psi_i(S_T) = (S_T - K_i)^+$and hence

$$\Psi_i(K_j) = (K_j - K_i)^+ .$$

Static Arbitrage Bounds

In this case, (4.6) can be written as

$$\mathbf{A}\mathbf{x} \geq \mathbf{0} \tag{4.7}$$

for

$$\mathbf{A} = \begin{bmatrix} K_2 - K_1 & 0 & 0 & \cdots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

This formulation is obtained by removing the first two constraints of (4.6) which are redundant in this particular case. Using this formulation, we obtain the following theorem giving necessary and sufficient conditions for a set of call option prices to contain no arbitrage opportunities.

Static Arbitrage Bounds

Theorem 4.4

Let $K_1 \leq K_2 \leq \cdots \leq K_n$ denote the strike prices of European call options written on the same underlying security with the same maturity. For $i = 1, \ldots, n$ let p_i denote the price of the ith call option. There are no arbitrage opportunities if and only if the prices $p_i, i = 1, \ldots, n$, satisfy the following conditions:

- $0 \leq p_n \leq p_{n-1} \leq \cdots \leq p_1$
- The piecewise linear function $C : [K_1, K_n] \to \mathbb{R}$ with breakpoints $K_1, \ldots, K_n$ defined by $C(K_i) := p_i, i = 1, \ldots, n$, is convex.

Static Arbitrage Bounds

The previous approach can be further extended to infer both lower and upper bounds on the current price p_{new} of a new derivative with maturity T and payoff $\Psi_{\text{new}}(S_T)$ given prices of other derivatives on the same underlying security and with the same maturity. As before, assume $\Psi_{\text{new}}(S_T)$ and $\Psi_i(S_T)$ are piecewise linear functions each with a single breakpoint K and $K_i, i = 1, \ldots, n$, respectively. Assume $K_1 \leq K_2 \leq \cdots \leq K_n$ and let p_i denote the current price of the i th derivative security.

Static Arbitrage Bounds

Assume there is no arbitrage involving the n derivatives with payoffs $\Psi_i(S_T)$, for $i = 1, \ldots, n$. The previous reasoning applied to the larger set of $n+1$ derivatives shows that there is no arbitrage if and only if the following two conditions hold:

- First, $p_{\text{new}} \geq \mathbf{p}^\top \mathbf{x}$ for any portfolio $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ such that

$$\Psi_{\text{new}}(S_T) \geq \Psi^{\mathbf{x}}(S_T) \text{ for all } S_T \geq 0$$

- Second, $p_{\text{new}} \leq \mathbf{p}^\top \mathbf{x}$ for any portfolio $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ such that

$$\Psi_{\text{new}}(S_T) \leq \Psi^{\mathbf{x}}(S_T) \text{ for all } S_T \geq 0.$$

Static Arbitrage Bounds

In words, the first condition states that the price of the new derivative has to be at least as large as the price of any sub-replicating portfolio of the old securities. Likewise, the second condition states that the price of the new derivative has to be at most as large as the price of any super-replicating portfolio of the old securities. The above two conditions automatically yield the following static arbitrage bounds on p.

Lower bound:

$$\begin{array}{rll} p^{\ell}_{\text{new}} := & \max & \mathbf{p}^{\top}\mathbf{x} \\ & \text{s.t.} & \Psi_{\text{new}}(S_T) \geq \Psi^{\mathbf{x}}(S_T) \text{ for all } S_T \geq 0 \end{array}$$

Upper bound:

$$\begin{array}{rll} p^{u}_{\text{new}} := & \min & \mathbf{p}^{\top}\mathbf{x} \\ & \text{s.t.} & \Psi_{\text{new}}(S_T) \leq \Psi^{\mathbf{x}}(S_T) \text{ for all } S_T \geq 0. \end{array}$$

Static Arbitrage Bounds

The piecewise linearity of $\Psi_{\text{new}}\ (S_T)$ and $\Psi_i\,(S_T)\,, i = 1, \ldots, n$, implies that both $\Psi^{\mathbf{x}}\,(S_T)$ for all $S_T \geq 0$, and $\Psi_{\text{new}}\ (S_T) \geq \Psi^{\mathbf{x}}\,(S_T)$ for lated as a finite system of linear inequalities. Therefore, : static arbitrage bounds can be formulated as linear proticular, for the special case where $\Psi_i\,(S_T) = (S_T - K_i)^+$, $\mathrm{w}\,(S_T) = (S_T - K)^+$with $K_1 \leq K \leq K_n$ the static arbion p can be written as the following linear programming

$$\begin{aligned} p^u_{\text{new}} \ := \min \quad & \mathbf{p}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned} \tag{4.8}$$

where

$$\mathbf{A} = \begin{bmatrix} K_2 - K_1 & 0 & 0 & \cdots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} (K_2 - K)^+ \\ (K_3 - K)^+ \\ \vdots \\ (K_n - K)^+ \\ 0 \end{bmatrix}.$$

Tax Clientele Effects in Bond Portfolio Management

This section presents a model proposed by Ronn (1987) to elicit clientele effects induced by taxes in the bond market. Related models were also proposed by Hodges and Schaefer (1977) and Schaefer (1982). The crux of the model is to formulate a linear program that exploits the price differential of bonds given their after-tax cash flows. To do so, the model finds a long–short portfolio that simultaneously buys “underpriced” bonds and sells “overpriced” bonds while ensuring non-negative cash flows throughout the lives of the bonds.

Tax Clientele Effects in Bond Portfolio Management

Next we describe the details of the model. Assume the bond market includes N bonds with the following characteristics:

- The ask and bid prices of bond j are p_j^a and p_j^b respectively for $j = 1, \dots, N$.
- Each unit of bond j generates a cash flow a_j^t at date t for $j = 1, \dots, N$ and $t = 1, \dots, T$. These cash flows are after-tax coupon and/or principal payments.
- The minimal risk-free reinvestment rate at future dates $t = 1, \dots, T$ is ρ.

Tax Clientele Effects in Bond Portfolio Management

Linear programming model for tax clientele effects in the bond market

Variables:

$$x_j^a: \text{ number of units of bond } j \text{ bought, for } j = 1, \ldots, N$$

$$x_j^b: \text{ number of units of bond } j \text{ sold, for } j = 1, \ldots, N$$

$$z_t: \text{ surplus cash flow at date } t, \text{ for } t = 1, \ldots, T.$$

Objective:

$$\max \sum_{j=1}^{N} p_j^b x_j^b - \sum_{j=1}^{N} p_j^a x_j^a.$$

Constraints:

Cash balance constraints in each date and bounds on x_j^a, x_j^b, and z_t :

$$
\begin{aligned}
z_1 &= \sum_{j=1}^{N} a_j^1 x_j^a - \sum_{j=1}^{N} a_j^1 x_j^b \\
z_t &= (1+\rho) z_{t-1} + \sum_{j=1}^{N} a_j^t x_j^a - \sum_{j=1}^{N} a_j^t x_j^b, \qquad \text{for } t = 2, \dots, T \\
x_j^a, x_j^b &\geq 0, \ \text{for } j = 1, \dots, N \\
z_t &\geq 0, \ \text{for } t = 1, \dots, T \\
x_j^a, x_j^b &\leq 1, \ \text{for } j = 1, \dots, N
\end{aligned}
\tag{4.9}
$$

The above objective function is the net difference between the value of the short positions and long positions of the portfolio. The short positions have to settle at the bid prices whereas the long positions have to settle at the ask prices. Because of this distinction, the constraints $x_j^a, x_j^b \geq 0$ are required. To ensure that the portfolio is risk-free, we require the surplus cash flows z_t to be non-negative for each date t.

The resulting linear program admits two main types of solutions. Either all bonds are priced within the bid-ask spread. In that case the optimal value of the linear program is zero and it is trivially attained by not taking any short or long positions. On the other hand, if there are exploitable price differentials in the bonds, the linear program chooses long and short holdings so as to maximize the difference between the values of the long and short positions. In that case the optimal value is positive. To avoid unbounded values, the model includes the upper bounds $x_j^a, x_j^b \leq 1$ on the long and short holdings.

Tax Clientele Effects in Bond Portfolio Management

Note that the model requires bonds with perfectly forecastable cash flows. Thus, non-callable bonds and notes are deemed appropriate, but callable bonds are excluded.

The proposed model explicitly accounts for the taxation of income and capital gains for specific investor classes. This means that the cash flows need to be adjusted for the presence of taxes. For a discount bond (that is, when $p_j^a < 100$), the after-tax cash flow of bond j at date t is

$$a_j^t = c_j^t(1-\tau),$$

where c_j^t is the coupon payment at date t and τ is the ordinary income tax rate. At maturity, the after-tax cash flow of bond j is

$$a_j^t = \left(100 - p_j^a\right)(1-g) + p_j^a,$$

where g is the capital gains tax rate.

Tax Clientele Effects in Bond Portfolio Management

On the other hand, for a premium bond (that is, when $p_j^a > 100$), the premium is amortized against ordinary income over the life of the bond, giving rise to an after-tax coupon payment of

$$a_j^t = \left[c_j^t - \frac{p_j^a - 100}{n_j} \right] (1 - \tau) + \frac{p_j^a - 100}{n_j},$$

where n_j is the number of coupon payments remaining to maturity. A premium bond also makes a non-taxable repayment of

$$a_j^t = 100$$

at maturity.

Tax Clientele Effects in Bond Portfolio Management

Major categories of taxable investors are domestic banks, insurance companies, individuals, non-financial corporations, and foreigners. In each case, one needs to distinguish the tax rates on capital gains versus ordinary income.

As an example, consider tax-exempt investors. For this class of investors, Schaefer (1982) observed that the "purchased" portfolio contains high coupon bonds and the "sold" portfolio is dominated by low coupon bonds. This can be explained as follows: The preferential taxation of capital gains for (most) taxable investors causes them to gravitate towards low coupon bonds. Consequently, for tax-exempt investors, low coupon bonds are "overpriced" and not desirable as investment vehicles.