HOMEWORK1

Exercise 2.3 Consider the linear programming problem

$$\begin{aligned} \max \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 14 \end{bmatrix}, \quad \mathbf{c}^\mathsf{T} = \begin{bmatrix} 6 & 5 & 4 & 3 & 5 & 4 \end{bmatrix}.$$

Solve this problem using the following strategy:

(a) Find the dual of the above primal linear program. The dual has only two variables. Solve the dual by inspection after drawing a graph of its feasible set.

(b) Using the optimal solution to the dual problem and the optimality conditions, determine what primal constraints are binding and what primal variables must be zero at an optimal solution. Using this information, determine the optimal solution to the primal linear program.

Exercise 2.8 Consider the following investment problem over T years, where the objective is to maximize the value of the investments in year T. We assume a perfect capital market with the same annual lending and borrowing rate $r > 0$ each year. We also assume that exogenous investment funds b_t are available in year t, for $t = 1, \ldots, T$. Let n be the number of possible investments. We assume that each investment can be undertaken fractionally (between 0 and 1). Let a_{tj} denote the cash flow associated with investment j in year t. Let c_j be the value of investment j in year T (including all cash flows subsequent to year T discounted at the interest rate r).

The linear program that maximizes the value of the investments in year T is the following. Denote by x_j the fraction of investment j undertaken, and let y_t be the amount borrowed (if negative) or lent (if positive) in year t:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j + y_T \\
s.t. \quad & -\sum_{j=1}^{n} a_{1j} x_j + y_1 \le b_1 \\
& -\sum_{j=1}^{n} a_{tj} x_j - (1+r) y_{t-1} + y_t \le b_t \quad && \text{for } t = 2, \ldots, T \\
& 0 \le x_j \le 1 && \text{for } j = 1, \ldots, n.
\end{aligned}
$$

(a) Write the dual of the above linear program.

(b) Solve the dual linear program found in part (a).

Hint: Note that some of the dual variables can be computed by backward substitution.

(c) Write the complementary slackness conditions.

(d) Deduce that the first T constraints in the primal linear program hold as equalities.

(e) Use the complementary slackness conditions to show that the solution obtained by setting $x_j = 1$ if $c_j + \sum_{t=1}^{T} (1+r)^{T-t} a_{tj} > 0$, and $x_j = 0$ otherwise, is an optimal solution.

(f) Conclude that the above investment problem always has an optimal solution where each investment is either undertaken completely or not at all.

Exercise 2.11 Use Theorem 2.4 to prove Theorem 2.5. To that end, proceed as follows.

(a) (Farkas's lemma) Consider the linear programming problem

$$\begin{aligned} \min \quad & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{0}. \end{aligned}$$

Show that the dual of this problem is

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Now apply Theorem 2.4.

(b) (Gordan's theorem) Proceed as in (a) but this time start with the linear programming problem

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \mathbf{A}^\mathsf{T}\mathbf{y} - \mathbf{1}t \geq \mathbf{0}. \end{aligned}$$

(c) (Stiemke's theorem) Proceed as in (a) and (b) but this time start with the linear programming problem

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{0} \\ & \mathbf{x} - \mathbf{1}t \geq \mathbf{0}. \end{aligned}$$

Exercise 3.1 You need to create a portfolio to cover the following stream of liabilities for the next six future dates:

Date	1	2	3	4	5	6
Required	500	200	800	200	800	1200

You may purchase the bonds in Table 3.2.

The term structure of risk-free interest rates is:

Date	1	2	3	4	5	6
Rate	5.04%	5.94%	6.36%	7.18%	7.89%	8.39%

(a) Formulate a linear programming model to find the lowest-cost long-only dedicated portfolio that covers the stream of liabilities with the bonds above. Assume surplus balances can be carried from one date to the next but earn no interest. What is the cost of your portfolio? What is the composition of your portfolio?

Table 3.2

Bond	Year 1	2	3	4	5	6	Price
1	10	10	10	10	10	110	109
2	7	7	7	7	7	107	94.8
3	8	8	8	8	8	108	99.5
4	6	6	6	6	106		93.1
5	7	7	7	7	107		97.2
6	6	6	6	106			96.3
7	5	5	5	105			92.9
8	10	10	110				110
9	8	8	108				104
10	6	6	106				101
11	10	110					107
12	7	107					102
13	100						95.2

(b) Formulate a linear programming model to find the lowest-cost portfolio that matches the present value and dollar duration of the stream of liabilities. What is the cost of your portfolio? How do the two present values change if interest rates decrease by one percentage point? How do they change if interest rates increase by one percentage point? How do they change if the interest rates in dates 1 and 2 decrease by one percentage point, the rates in dates 3, 4, and 5 remain the same, and the rate in date 6 increases by one percentage point?

(c) Use the linear programming sensitivity information from part (a) to determine the implied term structure of interest rates.

(d) Suppose that the stream of liabilities changes to:

Date	1	2	3	4	5	6
Required	100	200	800	500	800	1200

Find the new optimal dedicated portfolio and determine the new implied term structure. Is it different from the one you obtained in part (c)? Can you provide an intuitive explanation for the difference or lack thereof?