Homework2

Exercise 4.1 The Excel spreadsheet "Exercise 4.1 FX model" gives cross-currency exchange rates among the currencies USD, EUR, GBP, AUD, and JPY. Use a linear programming model to detect if these exchange rates contain an arbitrage opportunity. To do so, use the following decision variables:

x_{ij}: amount of currency i converted to currency j.

y_k: net amount of currency k after all transactions.

Is there an arbitrage opportunity? If the answer is yes, then describe it, for example: "Convert 1000 USD to EUR then to JPY then back to USD to net 1 USD without putting money in."

Exercise 4.3 Assume that the XYZ stock is currently priced at \$40. At the end of the next period, the price of XYZ is expected to be in one of the following two states: $40 \cdot u$ or $40 \cdot d$. We know that $d < 1 < \frac{5}{4} < u$ but we do not know d or u. The interest rate is zero. If a European call option with strike price \$50 is priced at \$10 while a European call option with strike price \$40 is priced at \$13, and we assume that these prices do not contain any arbitrage opportunities, what is the fair price of a European put option with a strike price of \$40?

Exercise 5.1 Assume $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is symmetric. Show that the function

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x}$$

is convex if and only if $\mathbf{Q}$ is positive semidefinite.

Assume $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite but not positive definite. Show that the problem

$$\min_{\mathbf{x}} \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x}$$

is either bounded or has infinitely many optimal solutions.

Exercise 6.1 The purpose of this exercise is to prove the two-fund theorem (Theorem 6.1).

(a) Find the Lagrangian function $L(\mathbf{x}, \theta)$ for (6.1).

(b) Solve the optimality conditions $\nabla L(\mathbf{x}, \theta) = \mathbf{0}$ to conclude that the optimal solution to (6.1) is

$$\mathbf{x}^* = \lambda \cdot \frac{1}{\mathbf{1}^\mathsf{T}\mathbf{V}^{-1}\boldsymbol{\mu}}\mathbf{V}^{-1}\boldsymbol{\mu} + (1-\lambda) \cdot \frac{1}{\mathbf{1}^\mathsf{T}\mathbf{V}^{-1}\mathbf{1}}\mathbf{V}^{-1}\mathbf{1}$$

where $\lambda = \mathbf{1}^\mathsf{T}\mathbf{V}^{-1}\boldsymbol{\mu}/\gamma$.

Exercise 6.5 The purpose of this exercise is to prove Proposition 6.4. Assume the covariance matrix of asset returns $\mathbf{V}$ is positive definite and the minimum- risk portfolio $(1/\mathbf{1}^\mathsf{T}\mathbf{V}^{-1}\mathbf{1})\mathbf{V}^{-1}\mathbf{1}$ has positive expected return; that is, $\boldsymbol{\mu}^\mathsf{T}\mathbf{V}^{-1}\mathbf{1} > 0$.

(a) Show that (6.19) can be rewritten as follows:

$$\begin{array}{lll} \min\limits_{\mathbf{z},\kappa} & \mathbf{z}^\mathsf{T}\mathbf{V}\mathbf{z} & \\ \text{s.t.} & \boldsymbol{\mu}^\mathsf{T}\mathbf{z} & = 1 \\ & \mathbf{1}^\mathsf{T}\mathbf{z} - \kappa & = 0 \\ & \kappa & > 0. \end{array} \tag{6.25}$$

(b) Show that the solution to (6.25) is

$$\mathbf{z}^* = \frac{1}{\boldsymbol{\mu}^\mathsf{T}\mathbf{V}^{-1}\boldsymbol{\mu}}\mathbf{V}^{-1}\boldsymbol{\mu}$$
$$\kappa^* = \mathbf{1}^\mathsf{T}\mathbf{z}^*.$$

(c) Use part (b) to conclude that the solution to (6.19) is indeed

$$\mathbf{x}^* = \frac{1}{\mathbf{1}^\intercal \mathbf{V}^{-1}\boldsymbol{\mu}}\mathbf{V}^{-1}\boldsymbol{\mu}.$$

(d) *Show that if $\boldsymbol{\mu}^\intercal \mathbf{V}^{-1}\mathbf{1} < 0$ then (6.19) is bounded but does not attain its maximum value. Use this fact to illustrate why the two assumptions made in Section 6.4 cannot simply be dropped without making some other assumptions.