Dynamic Programming Models:

the Binomial Pricing Model

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Introduction

One of the most common uses of dynamic programming in financial mathematics is through lattice models. In particular, the binomial lattice model of Cox et al. (1979) has become an indispensable tool for pricing derivative securities. This chapter describes this model and the underlying dynamic programming principles for the pricing of European options and the pricing and optimal exercising of American options.

Binomial Lattice Model

The binomial lattice provides a model for the price movements of a risky asset. It can be seen as a multi-period version of the single-period binomial model discussed in Section 4.3. The binomial lattice model describes the price of a risky asset at some discrete times $0, 1, \ldots, N$. A basic period length, such as a week, day, or second, is assumed to elapse between any two consecutive times. The model assumes that if the share price of the risky asset is S_k at time k then the share price S_{k+1} at time $k+1$ can take two values, namely $S_{k+1} = u \cdot S_k$ and $S_{k+1} = d \cdot S_k$ where $u > d > 0$ are multiplicative factors (u stands for "up" and d for "down" factors). The probabilities assigned to these two possible states are p and $1-p$ respectively, where $0 < p < 1$. The multi-stage price structure can be represented on a lattice as illustrated in Figure 15.1.

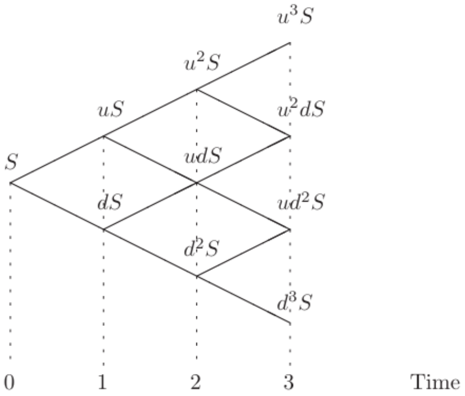

Figure 15.1: Asset price in the binomial lattice model

After k time periods, the asset price can take $k+1$ different values. If the price at stage 0 is S_0, then the price S_k at stage k is $u^j d^{k-j} S_0$ if there are j up moves and $k-j$ down moves. Observe that there are $\binom{k}{j}$ possible paths to reach the node corresponding to j up moves and $t-j$ down moves after t periods. Therefore the probability that the price is $u^j d^{k-j} S_0$ in stage k is $\binom{k}{j} p^j (1-p)^{k-j}$ because between two consecutive times the probability of an up move is p whereas that of a down move is $1-p$.

Option Pricing

Using the above binomial lattice model for the price process of an underlying risky asset, the value of an option on this asset can be computed by dynamic programming by using backward recursion, working from the maturity date (time N) back to time 0 (the current time). The approach fits within the stochastic setting introduced in Section 13.7. More precisely, the stages of the dynamic program are the discrete times $k = 0, \ldots, N$. The state at stage k is the asset price S_k. Thus S_k can take the $k+1$ possible values defined by the $k+1$ nodes in the k th layer of the lattice. The state S_N at the final stage N is the terminal state. The law of motion for the asset price is as follows:

$$S_{k+1} = \begin{cases} uS_k & \text{with probability } p \\ dS_k & \text{with probability } q = 1 - p \end{cases}$$

for $k = 0, 1, \ldots, N$. However, the following adjustment of utmost importance must be made: for option pricing purposes, we do not use these actual probabilities p and $q = 1 - p$ but instead the risk-neutral probabilities $\tilde{p}$ and $\tilde{q} = 1 - \tilde{p}$ as explained below.

European Options

Consider a European option contract that matures at time N with payoff $g(S_N)$ for some function $g(\cdot)$ of the underlying asset price S_N. For instance, if the contract is a European call option maturing at time N with strike price K, the payoff at maturity is $g(S_N) = (S_N - K)^+$. Similarly, if the contract is a European put option maturing at time N with strike price K, the payoff at maturity is $g(S_N) = (K - S_N)^+$.

Let $V_k(S_k)$ denote the value of the option at stage k when the asset price is S_k. This is the value-to-go function in our dynamic program. The value of the option at stage 0 is given by $V_0(S_0)$. This is the quantity that we have to compute in order to solve the option pricing problem. At the final time N the value-to-go function is given by the payoff of the option contract. That is,

$$V_N(S_N) = g(S_N)$$

Since we are dealing with a European option, we can compute the value $V_k(\cdot)$ in terms of $V_{k+1}(\cdot)$. The single-period subproblem between stages k and $k+1$ is identical to the single-period binomial model discussed in Section 4.3. Therefore, the value $V_k\left(S_k\right)$ can be obtained via the risk-neutral probabilities (4.4), namely

$$\tilde{p} = \frac{1+r-d}{u-d} \text{ and } \tilde{q} = \frac{u-1-r}{u-d}$$

where r is the one-period return on the risk-free asset between time k and time $k+1$. Thus for European options the value-to-go functions $V_k(\cdot)$ can be recursively computed as

$$V_k\left(S_k\right) = \frac{1}{1+r}\left(\tilde{p}V_{k+1}\left(uS_k\right) + \tilde{q}V_{k+1}\left(dS_k\right)\right). \tag{15.1}$$

Example 15.1

Consider a binomial lattice model with $N = 3, u = 2, d = \frac{1}{2}, r = 0.25$, and $S_0 = 40$ as depicted in Figure 15.2. Compute the price of a European call option with strike price $K = 50$.

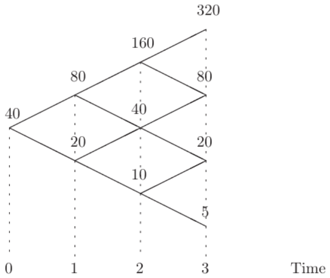

Figure 2: Binomial lattice with $u = 2, d = 0.5, S_0 = 40, N = 3$

Example 15.1

For these values of u, d, r the risk-neutral probabilities are

$$\tilde{p} = \tilde{q} = \frac{0.75}{1.5} = \frac{1}{2}$$

The option value $V_3(S_3) = (S_3 - 50)^+$ at the final stage 3 is as follows:

$$V_3(5) = V_3(20) = 0, \quad V_3(80) = 30, \quad V_3(320) = 270.$$

Next, applying (15.1) we get $V_2(S_2)$:

$$V_2(10) = 0, V_2(40) = \frac{1}{1.25} \cdot \frac{30}{2} = 12, V_2(160) = \frac{1}{1.25} \cdot \frac{30 + 270}{2} = 120.$$

Applying (15.1) again we get $V_1(S_1)$:

$$V_1(20) = \frac{1}{1.25} \cdot \frac{12}{2} = 4.8, V_1(80) = \frac{1}{1.25} \cdot \frac{120 + 12}{2} = 52.8$$

Finally applying (15.1) one more time, we get the option value $V_0(S_0)$:

$$V_0(40) = \frac{1}{1.25} \cdot \frac{4.8 + 52.8}{2} = 23.04$$

Example 15.2

Consider a binomial lattice model with $N = 3, u = 2, d = \frac{1}{2}$, $r = 0.25$, and $S_0 = 40$. Compute the price of a European put option with strike price $K = 60$. Again the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. We can proceed as in Example 15.1. The option value $V_3(S_3) = (60 - S_3)^+$at the final stage 3 is

$$V_3(5) = 55, V_3(20) = 40, \quad V_3(80) = V_3(320) = 0.$$

Next, applying (15.1) we get $V_2(S_2)$:

$$V_2(10) = \frac{1}{1.25} \cdot \frac{55 + 40}{2} = 38, V_2(40) = \frac{1}{1.25} \cdot \frac{40}{2} = 16, V_2(160) = 0$$

Applying (15.1) again we get $V_1(S_1)$:

$$V_1(20) = \frac{1}{1.25} \cdot \frac{38 + 16}{2} = 21.6, V_1(80) = \frac{1}{1.25} \cdot \frac{16}{2} = 6.4.$$

Finally applying (15.1) one more time, we get the option value $V_0(S_0)$:

$$V_0(40) = \frac{1}{1.25} \cdot \frac{21.6 + 6.4}{2} = 11.2.$$

American Options

Consider now an American option contract that can be exercised at any time $k = 0, 1, \ldots, N$ with payoff $g(S_k)$ for some function $g(\cdot)$ of the underlying asset price S_k. The key difference between this type of American option contract and the above type of European option contract is the possibility of early exercise. Because of this additional feature in the contract, the pricing problem of an American option needs to account for the optimal exercise timing of the option. This is accomplished via an adjustment to the previous recursion in the calculation of the value function.

Once again, at the final time N the value-to-go function is given by the payoff of the option contract. That is,

$$V_N(S_N) = g(S_N)$$

The computation of the value-to-go function $V_k(\cdot)$ in terms of $V_{k+1}(\cdot)$ needs to reflect the possibility of early exercise. To that end, the recursive formula (15.1) needs to be amended as follows:

$$V_k\left(S_k\right) = \max\left\{\frac{1}{1+r}\left(\tilde{p}V_{k+1}\left(uS_k\right) + \tilde{q}V_{k+1}\left(dS_k\right)\right), g\left(S_k\right)\right\}$$

In words, the value of the option at stage k is the maximum of the following two quantities: the first one is the discounted value of the option at stage $k+1$ or the payoff obtained if the option is exercised immediately. When the latter is larger, it is optimal to exercise the option at stage k.

Example 15.3

Consider a binomial lattice model with $N = 3, u = 2, d = \frac{1}{2}$, $r = 0.25$, and $S_0 = 40$. Compute the price of an American call option with strike price $K = 50$. Again the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. The option value $V_3(S_3) = (S_3 - 50)^+$at the final stage 3 is

$$V_3(5) = V_3(20) = 0, \quad V_3(80) = 30, V_3(320) = 270.$$

Next, applying (15.2) we get $V_2(S_2)$:

$$V_2(10) = 0, V_2(40) = \max\left\{\frac{1}{1.25}\cdot\frac{30}{2}, (40-50)^+\right\} = 12,$$
$$V_2(160) = \max\left\{\frac{1}{1.25}\cdot\frac{30+270}{2}, (160-50)^+\right\} = 120.$$

Example 15.3

Observe that regardless of the value of S_2, it is not optimal to exercise the option at stage 2.
Applying (15.2) again we get $V_1(S_1)$:

$$V_1(20) = \max\left\{\frac{1}{1.25}\cdot\frac{12}{2}, (20-50)^+\right\} = 4.8$$
$$V_1(80) = \max\left\{\frac{1}{1.25}\cdot\frac{120+12}{2}, (80-50)^+\right\} = 52.8$$

Observe that regardless of the value of S_1, it is not optimal to exercise the option at stage 1 .
Finally applying (15.2) one more time, we get the option value $V_0(S_0)$:

$$V_0(40) = \max\left\{\frac{1}{1.25}\cdot\frac{52.8+4.8}{2}, (40-50)^+\right\} = 23.04$$

Observe that there is no difference in price and in exercise policy between the American and the European call options.

Example 15.4

Consider a binomial lattice model with $N = 3, u = 2, d = \frac{1}{2}$, $r = 0.25$, and $S_0 = 40$. Compute the price of an American put option with strike price $K = 60$. Once again, the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. The option value $V_3(S_3) = (60 - S_3)^+$at the final stage 3 is

$$V_3(5) = 55, V_3(20) = 40, \quad V_3(80) = V_3(320) = 0$$

Next, applying (15.2) we get $V_2(S_2)$

$$V_2(10) = \max\left\{\frac{1}{1.25} \cdot \frac{55+40}{2}, (60-10)^+\right\} = 50$$

$$V_2(40) = \max\left\{\frac{1}{1.25} \cdot \frac{40}{2}, (60-40)^+\right\} = 20, V_2(160) = 0$$

Example 15.4

Observe that it is optimal to exercise the option at stage 2 when $S_2 = 10$ and $S_2 = 40$.
Applying (15.2) again we get $V_1(S_1)$

$$V_1(20) = \max\left\{\frac{1}{1.25}\cdot\frac{50+20}{2}, (60-20)^+\right\} = 40$$

$$V_1(80) = \max\left\{\frac{1}{1.25}\cdot\frac{20}{2}, (60-80)^+\right\} = 8$$

Observe that it is optimal to exercise the option at stage 1 when $S_1 = 20$. Finally applying (15.2) one more time, we get the option value $V_0(S_0)$

$$V_0(40) = \max\left\{\frac{1}{1.25}\cdot\frac{40+8}{2}, (60-40)^+\right\} = 20.$$

Observe that it is optimal to exercise the option at this stage.

Notice that the prices of the American and European calls in Example 15.1 and Example 15.3 are identical. This happens because there is nothing to gain by early exercising of the American call. By contrast, there is a substantial difference in the prices of the American and European puts in Example 15.2 and Example 15.4. This happens because sometimes it is advantageous to exercise an American put early. The results of these examples illustrate the following far more general property of American options.

Theorem 15.5

Theorem (15.5)

Consider the binomial lattice model described in Section 15.1, and an American option contract on the underlying risky asset that can be exercised at any time $k = 0, 1, \ldots, N$ *with payoff* $g(S_k)$ *for some function* $g(\cdot)$*. If* $r \geq 0$ *and the function* $g(\cdot)$ *is convex and satisfies* $g(0) = 0$*, then the value of the American option contract is the same as that of a European option contract with payoff* $g(S_N)$ *that can only be exercised at stage N. In other words, early exercising of the American option yields no advantage.*

Proof: By (15.2), it suffices to show that the following equation and inequality hold for $k = 0, 1, \ldots, N-1$:

$$V_k(S_k) = \frac{1}{1+r}\left(\tilde{p}V_{k+1}(uS_k) + \tilde{q}V_{k+1}(dS_k)\right) \geq g(S_k)$$

First, observe that for $k = 0, 1, \ldots, N-1$

$$S_k = \frac{1}{1+r}\left(\tilde{p}uS_k + \tilde{q}dS_k\right),$$

since $\tilde{p}, \tilde{q}$ are the risk-neutral probabilities. Next, we prove (15.3) by (backward) induction on k. The assumptions on the function $g(\cdot)$, equation (15.4), and $V_N(S_N) = g(S_N)$ imply that

$$\begin{aligned} g(S_{N-1}) &= g\left(\frac{r \cdot 0 + \tilde{p}uS_{N-1} + \tilde{q}dS_{N-1}}{1+r}\right) \\ &\leq \frac{r}{1+r} \cdot g(0) + \frac{1}{1+r}\left(\tilde{p}g(uS_{N-1}) + \tilde{q}g(dS_{N-1})\right) \\ &= \frac{1}{1+r}\left(\tilde{p}V_N(uS_{N-1}) + \tilde{q}V_N(dS_{N-1})\right) \end{aligned}$$

Therefore (15.3) holds for $k = N - 1$. Suppose (15.3) holds for $k = j + 1 \leq N - 1$. The assumptions on $g(\cdot)$, equation (15.4), and the induction hypothesis imply that

$$\begin{aligned} g(S_j) &= g\left(\frac{r \cdot 0 + \tilde{p}uS_j + \tilde{q}dS_j}{1+r}\right) \\ &\leq \frac{r}{1+r} \cdot g(0) + \frac{1}{1+r}\left(\tilde{p}g(uS_j) + \tilde{q}g(dS_j)\right) \\ &\leq \frac{1}{1+r}\left(\tilde{p}V_{j+1}(uS_j) + \tilde{q}V_{j+1}(dS_j)\right). \end{aligned}$$

Hence (15.3) holds for $k = j$ as well.

Option Pricing in Continuous Time

The binomial lattice model can be seen as a discrete version of a popular continuous-time geometric Brownian motion model. We next sketch some of the main ideas and results of this continuous model and its relation to the binomial lattice model discussed above. A full treatment of this topic is beyond the scope of this book. We refer the reader to Shreve (2000) for a detailed exposition of this topic.

Suppose the continuous-time price S_t, with $t \in [0, T]$, of a risky asset evolves according to the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where μ and σ are constants representing the instantaneous drift and volatility of the asset price S_t, and W_t is a Brownian motion.

The stochastic differential equation (15.5) can be seen as a continuous-time analog of the one-period up or down price movement in the binomial lattice model. The solution to (15.5) is the continuous-time process

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}$$

which can equivalently be written as

$$\log \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

Techniques from stochastic calculus have led to the development of pricing models for a wide variety of options provided the underlying risky asset is modeled via a suitable stochastic differential equation. In particular, in their seminal and ground-breaking work Black and Scholes (1973) and Merton (1973) derived a pricing formula for a European option on an underlying risky asset with a price process modeled as a geometric Brownian motion. In particular, consider a call option maturing at time $T > 0$ with payoff $(S_T - K)^+$.

Assume the price of the underlying risky asset is as in (15.6) and the risk-free asset compounds continuously at an instantaneous rate $r \geq 0$; that is, the price B_t of the risk-free asset is

$$B_t = B_0 e^{rt}.$$

The Black-Scholes-Merton model yields the following explicit formula for the price $V_t(S_t)$ at time $t \in [0, T]$ of a European call option with payoff $(S_T - K)^+$:

$$V_t(S_t) = \Phi(d_1) S_t - \Phi(d_2) K e^{-r(T-t)}$$

where

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right],$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The Black-Scholes-Merton model also yields the following formula for the price $V_t(S_t)$ at time $t \in [0, T]$ of a European put option with payoff $(K - S_T)^+$:

$$V_t(S_t) = \Phi(-d_2) K e^{-r(T-t)} - \Phi(-d_1) S_t,$$

where $\Phi(\cdot), d_1, d_2$ are the same as above.

The binomial lattice model can be seen as a discrete approximation of the geometric Brownian motion. The following section and Exercise 15.4 at the end of the chapter elaborate on this approximation.

Specifying the Model Parameters

To specify the binomial lattice model, one needs to choose values for u, d, and p. This is done by matching the mean and volatility of the asset price to the mean and volatility of the above binomial distribution. Because the model is multiplicative (the price S of the asset being either $u \cdot S$ or $d \cdot S$ in the next stage), it is convenient to work with $\log(S_{k+1}/S_k)$.

Let S_k denote the asset price in stages $k = 0, \ldots, N$. Let μ and σ be the mean and volatility of $\ln(S_N/S_0)$. (We assume that this information about the asset is known.) Let $\Delta = 1/N$ denote the length between consecutive stages. Then for $k = 0, 1, \ldots, N-1$ the mean and volatility of $\ln(S_{k+1}/S_k)$ are $\mu\Delta$ and $\sigma\sqrt{\Delta}$ respectively. In the binomial lattice, a direct computation shows that for $k = 0, 1, \ldots, N-1$ the mean and variance of $\ln(S_{k+1}/S_k)$ are $p \ln u + (1-p) \ln d$ and $p(1-p)(\ln u - \ln d)^2$ respectively. Matching these values we get two equations:

$$p \ln u + (1-p) \ln d = \mu\Delta$$
$$p(1-p)(\ln u - \ln d)^2 = \sigma^2\Delta$$

Note that there are three parameters but only two equations, so we can set $d = 1/u$ as in Cox et al. (1979). Then the equations simplify to

$$(2p - 1) \ln u = \mu\Delta$$
$$4p(1 - p)(\ln u)^2 = \sigma^2\Delta$$

Squaring the first and adding it to the second, we get $(\ln u)^2 = \sigma^2\Delta + (\mu\Delta)^2$. This yields

$$u = e^{\sqrt{\sigma^2\Delta + (\mu\Delta)^2}}$$
$$d = e^{-\sqrt{\sigma^2\Delta + (\mu\Delta)^2}}$$
$$p = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \sigma^2/\mu^2\Delta}}\right)$$

When Δ is small, these values can be approximated as

$$u \approx e^{\sigma\sqrt{\Delta}}$$
$$d \approx e^{-\sigma\sqrt{\Delta}}$$
$$p \approx \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right)$$

In other words, for small Δ

$$\log \frac{S_{k+1}}{S_k} \approx \begin{cases} \sigma\sqrt{\Delta} \text{ with probability } \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right) \\ -\sigma\sqrt{\Delta} \text{ with probability } \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta}\right) \end{cases}$$

which is a discrete approximation of (15.5).
As an example, consider a binomial model with 52 periods of one week each. Consider also a stock with current known price S_0 and random price S_{52} a year from today. We are given the mean μ and volatility σ of $\ln(S_{52}/S_0)$, say $\mu = 10\%$ and $\sigma = 30\%$. What are the parameters u, d, and p of the binomial lattice? Since $\Delta = \frac{1}{52}$ is small, we can use the second set of formulas:

$$u \approx e^{0.30/\sqrt{52}} = 1.0425$$

$$d \approx e^{-0.30/\sqrt{52}} = 0.9592$$

$$p \approx \frac{1}{2}\left(1 + \frac{0.10}{0.30\sqrt{52}}\right) = 0.523$$