Stochastic Programming Models: Asset–Liability Management

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Asset–Liability Management

The financial health of any company, and in particular of financial institutions, is reflected in the balance sheet of the company. Proper management of the company requires attention to both sides of the balance sheet - assets and liabilities. Asset-liability management offers sophisticated mathematical tools for an integrated management of assets and liabilities.

Asset-liability management recognizes that static, one-period investment planning models (such as mean-variance optimization) fail to incorporate the multiperiod nature of the liabilities faced by the company. A multi-period model that emphasizes the need to meet liabilities in each period for a finite (or possibly infinite) horizon is often required. Since liabilities and asset returns usually have random components, their optimal management requires techniques to optimize under uncertainty. In particular, stochastic programming approaches have been effective for these kinds of problems.

The main components of the asset-liability management problem are the stream of (random) liabilities faced by the firm, spread out over time, and the (random) returns of the assets that the firm may use for investments. Positions can be adjusted at each intermediate stage, adapting to the information revealed up to that stage. This is closely related to the financial planning example presented in Example 16.1.
The model assumes a planning horizon of T periods. Let $R_{i,t}$ denote the gross return of asset i between time $t-1$ and t, for $i=1,\ldots,n$ and $t=1,\ldots,T$. Let L_t denote the liability at time $t=1,\ldots,T$. Suppose we want to maximize the expected wealth of the firm at time T.

Multi-stage stochastic programming formulation

Variables:

$x_{i,t}$: amount invested in asset i at time t, for $i = 1, \ldots, n$ and $t = 0, 1, \ldots, T-1$.

Objective:

$$
\begin{array}{ll}
\max & \mathbb{E}\left[\sum_{i=1}^{n} R_{i,T} x_{i,T-1} - L_T\right] \\
\text{s.t.} & \sum_{i=1}^{n} R_{i,t} x_{i,t-1} = L_t + \sum_{i=1}^{n} x_{i,t}, \ \text{for } t = 1, \ldots, T-1 \\
 & x_{i,t} \geq 0, \ \text{for } i = 1, \ldots, n, t = 1, \ldots, T-1.
\end{array}
$$

The equality constraint in this formulation states that the surplus left after liability L_t is covered will be invested in the amounts $x_{i,t}$ in asset i for $i = 1, \ldots, n$.

The objective selected in the model above is to maximize the expected wealth at the end of the planning horizon. In practice, one might have a different objective. For example, in some cases, minimizing value at risk (VaR) or conditional value at risk (CVaR) might be more appropriate. Other priorities may dictate other objective functions.

To address the issue of the most appropriate objective function, one must understand the role of liabilities. Pension funds and insurance companies are among the most typical arenas for the integrated management of assets and liabilities.

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We consider the case of a Japanese insurance company, the Yasuda Fire and Marine Insurance Co. Ltd., following the work of Cariño et al. (1994). In this case, the liabilities are mainly savings-oriented policies issued by the company. Each new policy sold represents a deposit, or inflow of funds. Interest is periodically credited to the policy until maturity, typically three to five years, at which time the principal amount plus credited interest is refunded to the policyholder. The crediting rate is typically adjusted each year in relation to a market index like the prime rate. Therefore, we cannot say with certainty what the future liabilities will be. Insurance business regulations stipulate that interest credited to some policies be earned from investment income, not capital gains. So, in addition to ensuring that the maturity cash flows are met, the firm must seek to avoid interim shortfalls in income earned versus interest credited. In fact, it is the risk of not earning adequate income quarter by quarter that the decision makers view as the primary component of risk at Yasuda.

The problem is to determine the optimal allocation of the deposited funds into several asset categories: cash, fixed-rate and floating-rate loans, bonds, equities, real estate, and other assets. Since we can revise the portfolio allocations over time, the decision we make is not just among allocations today but among allocation strategies over time. A realistic dynamic asset-liability model must also account for the payment of taxes. This is made possible by distinguishing between interest income and price return.

A stochastic linear program is used to model the problem. The linear program has uncertainty in many coefficients. This uncertainty is modeled through a finite number of scenarios. In this fashion, the problem is transformed into a very large-scale linear program of the form (16.3). The random elements include price return and interest income for each asset class, as well as policy crediting rates. We next describe the main components of the multi-stage stochastic programming model.

Main Components

Stages: The stages of the model are indexed by $t = 0, 1, \ldots, T$.
Variables: $x_{i,t}$ = market value in asset i at stage t for $i = 1, \ldots, n$ and $t = 0, 1, \ldots, T$.
w_t = interest income shortfall at stage for $t = 1, \ldots, T$.
v_t = interest income surplus at stage for $t = 1, \ldots, T$.

Random parameters in the stochastic linear program:
$RP_{i,t}$ = price return of asset i between stage $t-1$ and stage t, for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.
$RI_{i,t}$ = interest income of asset i between stage $t-1$ and stage t, for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.
F_t = deposit inflow between stage $t-1$ and stage t, for $t = 1, \ldots, T$.
P_t = principal payout between stage $t-1$ and stage t, for $t = 1, \ldots, T$.
I_t = interest payout between stage $t-1$ and stage t, for $t = 1, \ldots, T$.
g_t = rate at which interest is credited to policies between stage $t-1$ and stage t, for $t = 1, \ldots, T$.
L_t = liability valuation at stage t.

Parameterized objective function components:
$c_t(\cdot)$ = piecewise linear convex penalty for shortfall at time t.

The goal of the model is to allocate funds among available assets to maximize expected wealth at the end of the planning horizon T minus the expected penalized shortfall accumulated through the planning horizon. The problem can be formulated as the following multi-stage stochastic program:

$$
\begin{aligned}
\max \quad & \mathbb{E}\left[\sum_{i=1}^{n} x_{i,T} - \sum_{t=1}^{T} c_t\left(w_t\right)\right] \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{i,t} - \sum_{i=1}^{n} \left(1 + RP_{i,t} + RI_{i,t}\right) x_{i,t-1} = F_t - P_t - I_t \quad \text{for } t = 1 \\
& \text{ asset accumulation} \\
& \sum_{i=1}^{n} RI_{i,t} x_{i,t-1} + w_t - v_t = g_t L_{t-1} \quad \text{for } t = 1, \ldots, T \\
& \text{interest income shortfall} \\
& L_t = \left(1 + g_t\right) L_{t-1} + F_t - P_t - I_t \quad \text{for } t = 1, \ldots, T \\
& \text{liability accumulation} \\
& x_{i,t} \geq 0, \quad w_t \geq 0, \quad v_t \geq 0.
\end{aligned}
\tag{17.1}
$$

In the model discussed in Cariño et al. (1994), the stochastic linear program (17.1) is converted into a large linear program using a finite number of scenarios to deal with the random elements in the data. Creation of scenario inputs is made in stages using a tree. The tree structure can be described by the number of branches at each stage. For example, a 1-8-4-4-2-1 tree has 256 scenarios. Stage $t = 0$ is the initial stage. Stage $t = 1$ may be chosen to be the end of Quarter 1 and has eight different branches in this example. Stage $t = 2$ may be chosen to be the end of Year 1, with each of the previous eight branches giving rise to four new branches, and so on. For the Yasuda Fire and Marine Insurance Co. Ltd., a problem with seven asset classes and six stages gives rise to a stochastic linear program (17.1) with 12 constraints (other than non-negativity) and 54 variables. Using 256 scenarios, this stochastic program is converted into a linear program with several thousand constraints and over 10,000 variables. Solving this model yielded extra income estimated to be about US $80 million per year for the company.

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The Option Pricing Problem

The option pricing problem discussed in Chapter 15 and modeled via the binomial lattice can alternatively be formulated as a stochastic programming problem. As should be expected, the two approaches are equivalent under the assumptions made for the binomial lattice model. However, there is additional flexibility in the stochastic programming approach that makes it applicable under less restrictive assumptions. In particular, we will discuss how the stochastic programming approach can easily model transaction costs. This is an important practical issue that cannot be incorporated in the binomial lattice model.

We will work with the following similar setting to that in Chapter 15. Let S_t, for $t = 0, 1, \ldots, N$, denote the share price of a risky asset at times $t = 0, 1, \ldots, N$. Assume the economy also has a risk-free asset whose interest rate is r in each period$[t-1, t]$ for $t = 1, \ldots, N$.

European Options

Consider a European option contract that matures at time N with payoff $g(S_N)$ for some function $g(\cdot)$ of the underlying asset price S_N. The following stochastic program provides a model for the lowest-cost portfolio of the underlying asset and the risk-free asset that can be constructed at time 0 and be subsequently rebalanced to super-replicate the payoff $g(S_N)$ of the European option contract.

Variables:

x_t = amount of shares of the risky asset at time t for $t = 0, \ldots, N-1$.

y_t = amount of money in the risk-free asset at time t for $t = 0, \ldots, N-1$.

Objective:

$$\begin{aligned} & \min S_0 x_0 + y_0 \\ & \text{s.t.} S_N x_{N-1} + (1+r) y_{N-1} \geq g(S_N) \\ & \quad S_t x_{t-1} + (1+r) y_{t-1} \geq S_t x_t + y_t, \quad t = 1, \ldots, N-1. \end{aligned} \tag{17.2}$$

Consider the following binomial tree model for the risky prices. Assume that there are exactly two possible random outcomes (" H " and " T ") between time $t-1$ and t for $t=1,\ldots,N$. For the simplest case $N=1$, the binomial tree model yields the following deterministic equivalent of (17.2):

$$\begin{aligned} & \min S_0 x_0 + y_0 \\ \text{s.t. } & S_1(H)x_0 + (1+r)y_0 \geq g\left(S_1(H)\right) \\ & S_1(T)x_0 + (1+r)y_0 \geq g\left(S_1(T)\right) \end{aligned} \tag{17.3}$$

Observe that the linear programming dual of (17.3) is

$$\begin{aligned} & \max g\left(S_1(H)\right) v(H) + g\left(S_1(T)\right) v(T) \\ \text{s.t. } & S_1(H)v(H) + S_1(T)v(T) = S_0 \\ & (1+r)v(H) + (1+r)v(T) = 1 \\ & v(H), v(T) \geq 0 \end{aligned} \tag{17.4}$$

which in turn can be rewritten via the change of variables $\tilde{p} := (1+r)v(H)$ and $\tilde{q} := (1+r)v(T)$ as

$$
\begin{aligned}
\max & \frac{1}{1+r}\left(g\left(S_1(H)\right)\tilde{p} + g\left(S_1(T)\right)\tilde{q}\right) \\
\text{s.t. } & \frac{1}{1+r}\left(S_1(H)\tilde{p} + S_1(T)\tilde{q}\right) = S_0 \\
& \tilde{p} + \tilde{q} = 1 \\
& \tilde{p}, \tilde{q} \geq 0
\end{aligned}
\tag{17.5}
$$

Without loss of generality assume $S_1(H) \geq S_1(T)$. Furthermore, assume $S_1(H) > S_1(T)$ as otherwise the pricing problem of the option contract is trivial. It follows that (17.5) is feasible if and only if $S_1(T) \leq (1+r)S_0 \leq S_1(H)$.

In this case the only feasible solution to (17.5) is

$$\tilde{p} = \frac{(1+r)S_0 - S_1(T)}{S_1(H) - S_1(T)}, \quad \tilde{q} = \frac{S_1(H) - (1+r)S_0}{S_1(H) - S_1(T)},$$

and thus the optimal value of (17.3) and (17.4) is

$$\frac{1}{1+r}\left(\tilde{p}g\left(S_1(H)\right) + \tilde{q}g\left(S_1(T)\right)\right).$$

Observe that this price is exactly the same (as it should be) as the one obtained via the one-period binomial model discussed in Section 4.3 when the singleperiod economy has no arbitrage. A similar duality argument shows that in the absence of arbitrage the stochastic programming model (17.2) is equivalent to the binomial lattice approach for the general multi-period case; that is, when $N > 1$. The following example illustrates this equivalence.

Example

Suppose $n=2, r=\frac{1}{4}$, and the prices S_0, S_1, S_2 of the risky asset are as indicated at the nodes of the binomial tree depicted in Figure 17.1. Assume that the two branches emerging from each node are equally likely. Determine the price of a European put option maturing at time $N=2$ with strike price 50 ; that is, with payoff $g\left(S_2\right)=\left(50-S_2\right)^{+}$.
In this case the deterministic equivalent of (17.2) is

$$
\begin{array}{ll}
\min & 40x_0+y_0 \\
\text{s.t.} & 80x_0+1.25y_0 \geq 80x_1(H)+y_1(H) \\
& 20x_0+1.25y_0 \geq 20x_1(T)+y_1(T) \\
& 160x_1(H)+1.25y_1(H) \geq (50-160)^{+}=0 \\
& 40x_1(H)+1.25y_1(H) \geq (50-40)^{+}=10 \\
& 40x_1(T)+1.25y_1(T) \geq (50-40)^{+}=10 \\
& 10x_1(T)+1.25y_1(T) \geq (50-10)^{+}=40
\end{array}
$$

The optimal solution to this linear program is

$$
\begin{aligned}
x_1(H) = -0.0833, y_1(H) = 10.6666, x_1(T) = -1, y_1(T) = 40, (17.6) \\
x_0 = -0.2666, y_0 = 20.2666
\end{aligned}
$$

and thus its optimal value is 9.6. On the other hand, the binomial lattice approach would yield the risk-neutral probabilities $\tilde{p} = \tilde{q} = \frac{1}{2}$. Consequently, the value $V_1(S_1)$ of the option at time 1 is

$$
V_1(80) = \frac{1}{1.25} \cdot \frac{0+10}{2} = 4, V_1(20) = \frac{1}{1.25} \cdot \frac{10+40}{2} = 20
$$

and the value $V_0(S_0)$ of the option at time 0 is

$$
V_0(40) = \frac{1}{1.25} \cdot \frac{4+20}{2} = 9.6
$$

The (super-)replicating portfolio (17.6) can also be recovered via delta-hedging.

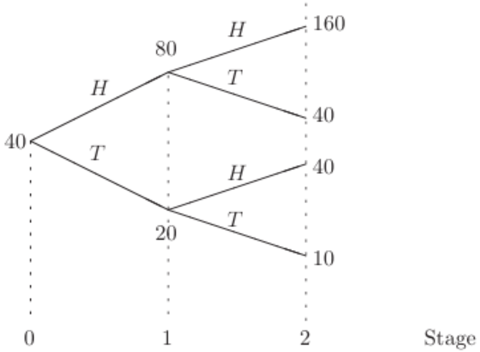

Figure: Binomial tree for option pricing example

American Options

Consider now an American option contract that can be exercised at any time $t = 0, 1, \ldots, N$ with payoff $g(S_t)$ for some function $g(\cdot)$ of the underlying asset price S_t. The stochastic program (17.2) has the following straightforward modification for finding a lowest-cost portfolio of the underlying asset and the risk-free asset that can be constructed at time 0 and be subsequently rebalanced to superreplicate the payoff of the American option contract:

$$
\begin{array}{ll}
\min & S_0 x_0 + y_0 \\
\text{s.t.} & S_N x_{N-1} + (1+r) y_{N-1} \geq g(S_N) \\
 & S_t x_{t-1} + (1+r) y_{t-1} \geq \max\{S_t x_t + y_t, g(S_t)\}, t = 1, \ldots, N-1.
\end{array}
$$

The latter problem in turn can be equivalently stated as follows:

$$
\begin{aligned}
&\min S_0 x_0 + y_0 \\
&\text{s.t.} S_N x_{N-1} + (1+r) y_{N-1} \geq g\left(S_N\right) \\
&\quad S_t x_{t-1} + (1+r) y_{t-1} \geq S_t x_t + y_t, t = 1, \ldots, N-1 \\
&\quad S_t x_{t-1} + (1+r) y_{t-1} \geq g\left(S_t\right), \quad t = 1, \ldots, N-1.
\end{aligned} \tag{17.7}
$$

Again for a binomial event tree model the above stochastic programming approach is equivalent to the binomial lattice approach discussed in Chapter 15 in the absence of arbitrage.

Transaction Costs

The stochastic programming models (17.2) and (17.7) can be readily extended to incorporate proportional transaction costs. Observe that in the absence of transaction costs a transaction to sell w shares of the risky asset when its price is S will generate a revenue equal to wS. By contrast, if a proportional transaction cost θ applies to the sell transaction then the revenue would instead be $(1-\theta)wS$. Similarly, if a proportional transaction cost θ applies to a buy transaction of w shares, then the cost of the transaction would be $(1+\theta)wS$.

The stochastic programming model (17.2) can be modified as follows to account for a proportional transaction cost θ applicable to each buy or sell transaction of the risky asset:

$$\begin{aligned} \min\; & S_0 x_0 + \theta \left| S_0 x_0 \right| + y_0 \\ \text{s.t.}\; & S_N x_{N-1} + (1+r) y_{N-1} - \theta \left| S_N x_{N-1} \right| \geq g\left(S_N\right) \\ & S_t x_{t-1} + (1+r) y_{t-1} - \theta \left| S_t \left(x_t - x_{t-1}\right) \right| \geq S_t x_t + y_t, \quad t = 1, \ldots, N - \end{aligned} \tag{17.8}$$

Similarly, the stochastic programming model (17.7) can also be modified to account for the same kind of transaction costs as follows:

$$
\begin{aligned}
\min & S_0 x_0 + \theta \left| S_0 x_0 \right| + y_0 \\
\text{s.t. } & S_N x_{N-1} + (1+r) y_{N-1} - \theta \left| S_N x_{N-1} \right| \geq g\left(S_N\right) \\
& S_t x_{t-1} + (1+r) y_{t-1} - \theta \left| S_t \left(x_t - x_{t-1}\right) \right| \geq S_t x_t + y_t, \quad t = 1, \ldots, N - \\
& S_t x_{t-1} + (1+r) y_{t-1} - \theta \left| S_t x_t \right| \geq g\left(S_t\right), \quad t = 1, \ldots, N-1 \quad (17.9)
\end{aligned}
$$

Observe that the stochastic programs (17.9) and (17.8) include some term with absolute values in the objective and constraints. The models can be recast as linear stochastic programs by introducing some extra variables and constraints, as the following example illustrates.

Example

Example 17.2
Suppose $n = 2, r = \frac{1}{4}$, and the prices S_0, S_1, S_2 of the risky asset are as indicated at the nodes of the binomial tree depicted in Figure 17.1. Assume that the two branches emerging from each node are equally likely. Determine the price of a European put option maturing at time $N = 2$ with strike price 50 ; that is, with payoff $g\left(S_2\right) = \left(50 - S_2\right)^+$. Assume a proportional transaction cost θ applies to every buy or sell transaction.

In this case the deterministic equivalent of (17.8) is

$$
\begin{array}{ll}
\min & 40x_0 + y_0 + 40\theta u_0 \\
\text{s.t.} & 80x_0 + 1.25y_0 - 80\theta v_1(H) \geq 80x_1(H) + y_1(H) \\
& 20x_0 + 1.25y_0 - 20\theta v_1(T) \geq 20x_1(T) + y_1(T) \\
& 160x_1(H) + 1.25y_1(H) - 160\theta w_1(H) \geq (50-160)^+ = 0 \\
& 40x_1(H) + 1.25y_1(H) - 40\theta w_1(H) \geq (50-40)^+ = 10 \\
& 40x_1(T) + 1.25y_1(T) - 40\theta w_1(T) \geq (50-40)^+ = 10 \\
& 10x_1(T) + 1.25y_1(T) - 10\theta w_1(T) \geq (50-10)^+ = 40 \\
& u_0 \geq x_0, \quad u_0 \geq -x_0 \\
& v_1(H) \geq x_1(H) - x_0, \quad v_1(T) \geq -x_1(T) + x_0 \\
& w_1(H) \geq x_1(H), \quad w_1(H) \geq -x_1(H) \\
& w_1(T) \geq x_1(T), \quad w_1(T) \geq -x_1(T).
\end{array}
$$

Table 17.1 shows the optimal value and holdings
$x_0, y_0, x_1(H), y_1(H), x_1(T), y_1(T)$ of
the optimal super-replicating portfolio for various levels of transaction cost θ.
Table 17.1

θ	Optimal value	x_0	y_0	$x_1(H)$	$y_1(H)$	$x_1(T)$	$y_1(T)$
0	9.6	-0.266	20.266	-0.083	10.666	-1	40
0.01	9.930	-0.268	20.574	-0.082	10.666	-0.990	40
0.05	11.243	-0.275	21.710	-0.079	10.666	-0.952	40
0.1	12.849	-0.280	22.948	-0.075	10.666	-0.909	40

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An important issue in portfolio selection is the potential decline of the portfolio value below some critical limit. How can we control the risk of downside losses? A possible answer is to create a payoff structure similar to a European call option.

While a corporate investor may be able to construct a diversified portfolio, there may be no option market available on this portfolio. One solution may be to use index options. However, exchange-traded options with sufficient liquidity are limited to maturities of about three months. This makes the cost of long-term protection expensive, requiring the purchase of a series of highly priced shortterm options. For large institutional or corporate investors, a cheaper solution is to artificially produce the desired payoff structure using available resources. This is called a synthetic option strategy. A model of this kind was proposed by Zhao and Ziemba (2001) and can be described as follows.

Problem parameters:

W_0 = investor's initial wealth

T = investor's planning horizon

R = gross return of a riskless asset for one period

$R_{i,t}$ = gross return for asset i at time t

$\theta_{i,t}$ = transaction cost for purchases and sales of asset i at time t.

The gross returns $R_{i,t}$ above are random, but their distributions are known.

Variables:

$x_{i,t}$ = amount allocated to asset i at time t

$A_{i,t}$ = amount of asset i bought at time t

$D_{i,t}$ = amount of asset i sold at time t

y_t = amount allocated to riskless asset at time t.

We formulate a stochastic program that produces the desired payoff at the end of the planning horizon T, much in the flavor of the stochastic programs developed in the previous section. Let us first discuss the constraints.

The initial portfolio must satisfy

$$y_0 + x_{1,0} + \ldots + x_{n,0} = W_0$$

Similarly, the portfolio at time t must satisfy

$$\begin{aligned} &x_{i,t} = R_{i,t}x_{i,t-1} + A_{i,t} - D_{i,t} && \text{for } t = 1, \ldots, T \\ &y_t = Ry_{t-1} - \sum_{i=1}^{n} (1 + \theta_{i,t}) A_{i,t} + \sum_{i=1}^{n} (1 - \theta_{i,t}) D_{i,t} && \text{for } t = 1, \ldots, T. \end{aligned}$$

One can also impose upper bounds on the proportion of any risky asset in the portfolio:

$$0 \leq x_{i,t} \leq m_t \left(y_t + \sum_{j=1}^{n} x_{j,t} \right)$$

where m_t is chosen by the investor.

The value of the portfolio at the end of the planning horizon is

$$v = Ry_{T-1} + \sum_{i=1}^{n} (1 - \theta_{i,T}) R_{i,T} x_{i,T-1}$$

where the summation term is the value of the risky assets at time T. To construct the desired synthetic option, we split v into the riskless value of the portfolio Z and a surplus $z \geq 0$ which depends on random events. Using a scenario approach to the stochastic program, Z is the worst-case payoff over all the scenarios. The surplus z is a random variable that depends on the scenario. Thus

$$v = Z + z, \quad z \geq 0$$

We consider Z and z as variables of the problem, and we optimize them together with the asset allocations x and other variables described earlier. The objective function of the stochastic program is

$$\max \mathbb{E}(z) + \mu Z$$

where $\mu \geq 1$ is the risk aversion of the investor. The risk aversion μ is given data. When $\mu = 1$, the objective is to maximize expected return. When μ is very large, the objective is to maximize "riskless profit".

Example

Example 17.3 an investor with initial wealth $W_0 = 1$ who wants to construct a portfolio comprising one risky asset and one riskless asset using the "synthetic option" model described above. We next describe the deterministic equivalent of this model for a two-period planning horizon, i.e. $T = 2$, and an event tree with four scenarios. The construction is similar to that in Example 16.2. Suppose the return on the riskless asset is a non-random value R per period and there are two equally likely possible random outcomes (" H " and " T ") over each time period. Let $R_t(H)$ and $R_t(T)$ denote the return of the risky asset in the period $[t-1, t]$ when the outcome is H and T respectively. Suppose the transaction cost for purchases and sales of the risky asset is a non-random value θ.

The scenario tree in this case is identical to that depicted in Figure 16.1 for Example 16.2. The deterministic equivalent of the multi-stage stochastic linear program in this case is as follows:

$$
\begin{array}{ll}
\max & \frac{1}{4}(z(HH)+z(HT)+z(TH)+z(TT))+\mu Z \\
\text{s.t.} & y_0+x_0=1 \\
& x_1(H)=R_1(H)x_0+A_1(H)-D_1(H) \\
& x_1(T)=R_1(T)x_0+A_1(T)-D_1(T) \\
& y_1(H)=Ry_0-(1+\theta)A_1(H)+(1-\theta)D_1(H) \\
& y_1(T)=Ry_0-(1+\theta)A_1(T)+(1-\theta)D_1(T) \\
& z(HH)+Z=Ry_1(H)+(1-\theta)R_2(H)x_1(H) \\
& z(HT)+Z=Ry_1(H)+(1-\theta)R_2(T)x_1(H) \\
& z(TH)+Z=Ry_1(T)+(1-\theta)R_2(H)x_1(T) \\
& z(TT)+Z=Ry_1(T)+(1-\theta)R_2(T)x_1(T) \\
& x_0,y_0,x_1(H),x_1(T),y_1(H),y_1(T),A_1(H),D_1(H),A_1(T),A_2(T)\geq 0 \\
& z(HH),z(HT),z(TH),z(TT)\geq 0
\end{array}
$$

Z free.

Zhao and Ziemba (2001) introduce and apply the above generic synthetic option model to an example with three assets (cash, bonds, and stocks) and four periods (a one-year horizon with quarterly portfolio reviews). The quarterly return on cash is constant at $\rho = 0.0095$. For stocks and bonds, the expected logarithmic rates of returns are $s = 0.04$ and $b = 0.019$ respectively. Transaction costs are assumed to be 0.5% for stocks and 0.1% for bonds. The scenarios needed in the stochastic program are generated using an autoregression model which is constructed based on historical data (quarterly returns from 1985 to 1998; the Salomon Brothers bond index and S&P 500 index respectively). Specifically, the autoregression model is

$$s_t = 0.037 - 0.193 s_{t-1} + 0.418 b_{t-1} - 0.172 s_{t-2} + 0.517 b_{t-2} + \epsilon_t$$
$$b_t = 0.007 - 0.140 s_{t-1} + 0.175 b_{t-1} - 0.023 s_{t-2} + 0.122 b_{t-2} + \eta_t$$

where the pair (ϵ_t, η_t) characterizes uncertainty. Zhao and Ziemba used a random sampling approach to estimate the joint distribution of (ϵ_t, η_t). From this joint distribution of (ϵ_t, η_t) a set of 20 pairs can be selected to estimate the empirical distribution of (ϵ_t, η_t). In this way, a scenario tree with $160,000(= 20 \times 20\times\ 20 \times 20)$ paths describing possible outcomes of asset returns is generated for the four periods.

The authors solved the resulting large deterministic linear program. We discuss some of the results obtained when this linear program is solved for a risk aversion of $\mu = 2.5$. The value of the terminal portfolio is always at least 4.6% more than the initial portfolio wealth and the distribution of terminal portfolio values is skewed to larger values because of dynamic downside risk control. The expected return is 16.33% and the volatility is 7.2%. It is interesting to compare these values with those obtained from a static Markowitz model. The expected return is 15.4% for the same volatility but no minimum return is guaranteed. In fact,in some scenarios, the value of the Markowitz portfolio is 5% less at the end of the one-year horizon than it was at the beginning.

It is also interesting to look at a typical portfolio (one of the 160,000 paths) generated by the synthetic option model (the linear program was set up with an upper bound of 70% placed on the fraction of stocks or bonds in the portfolio).

Quarter t	Cash	Stocks	Bonds	Portfolio value at the end of Quarter t
1	12%	18%	70%	103
2		41%	59%	107
3		70%	30%	112
4	30%		70%	114