Robust Optimization

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Robust optimization

In robust optimization, the description of the parameter uncertainty is formalized via uncertainty sets. Uncertainty sets can represent or may be formed by difference of opinions on the possible values of problem parameters, alternative estimates of parameters generated via statistical techniques from historical data, Bayesian, or other estimation techniques. The size of the uncertainty set is typically determined by the level of desired robustness.

Some of the most common types of uncertainty sets encountered in robust optimization models include the following:

- Uncertainty sets representing a finite number of scenarios generated for the possible values of the parameters:

$$\mathcal{U} = \{p_1, p_2, \dots, p_k\}$$

- Uncertainty sets representing the convex hull of a finite number of scenarios generated for the possible values of the parameters (these are sometimes called polytopic uncertainty sets):

$$\mathcal{U} = conv\{p_1, p_2, \dots, p_k\}$$

- Uncertainty sets representing an interval description for each uncertain parameter:

$$\mathcal{U} = \{p : 1 \leq p \leq u\}$$

  Confidence intervals encountered frequently in statistics can be the source of such uncertainty sets.

- Ellipsoidal uncertainty sets:

$$\mathcal{U} = \{p : p = p_0 + Mu, \|u\| \leq 1\}$$

It is a non-trivial task to determine the uncertainty set that is appropriate for a particular model as well as the type of uncertainty sets that lead to tractable problems. As a general guideline, the shape of the uncertainty set will often depend on the sources of uncertainty as well as the sensitivity of the solutions to these uncertainties. The size of the uncertainty set, on the other hand, will often be chosen based on the desired level of robustness. When uncertain parameters reflect the "true" values of moments of random variables, as is the case in mean–variance portfolio optimization, we simply have no way of knowing these unobservable true values exactly.

In such cases, after making some assumptions about the stationarity of these random processes, we can generate estimates of these true parameters using statistical procedures. Goldfarb and Iyengar (2003), for example, show that if we use a linear factor model for the multivariate returns of several assets and estimate the factor loading matrices via linear regression, the confidence regions generated for these parameters are ellipsoidal sets and they advocate their use in robust portfolio selection as uncertainty sets. To generate interval-type uncertainty sets, Tütüncü and Koenig (2004) use bootstrapping strategies as well as moving averages of returns from historical data. The shape and the size of the uncertainty set can significantly affect the robust solutions generated. However, with few guidelines backed by theoretical and empirical studies, their choice remains a mix of art and science.

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As we next describe, different types of robustness arise depending on what parameters of a problem are uncertain, and depending also on what exactly constitutes a "good" robust solution.

Constraint Robustness

Constraint robustness refers to situations where the uncertainty is in the constraints and we seek solutions that remain feasible for all possible values of the uncertain inputs. This type of solution is required in many engineering applications. Typical instances include multi-stage problems where the uncertain outcomes of earlier stages have an effect on the decisions of the later stages and the decision variables must be chosen to satisfy constraints no matter what happens with the uncertain parameters of the problem.

Here is a precise mathematical model for finding constraint-robust solutions. Consider an optimization problem of the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & G(\mathbf{x}, \mathbf{p}) \in K. \end{aligned} \tag{19.1}$$

In this problem x is the vector of decision variables, f(x) is the (certain) objective function, G and K are the structural elements of the constraints assumed to be certain, and p is the vector of possibly uncertain parameters of the problem. Consider an uncertainty set U that contains all possible values of the uncertain parameters p. Then, a constraint-robust optimal solution can be found by solving the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & G(\mathbf{x}, \mathbf{p}) \in K, \quad \text{for all} \quad \mathbf{p} \in \mathcal{U}. \end{aligned} \tag{19.2}$$

The feasible set in the robust optimization model (19.2) is the intersection of the feasible sets:

$$S(\mathbf{p}) = \{\mathbf{x} : G(\mathbf{x}, \mathbf{p}) \in K\}, \quad \mathbf{p} \in \mathcal{U}.$$

We note that there are no uncertain parameters in the objective function of the problem (19.1). However, this is not a restrictive assumption. An optimization problem with uncertain parameters in both the objective function and constraints can be easily reformulated to fit the form in (19.1). Indeed, the problem

$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}, \mathbf{p}) \\ \text{s.t.} & G(\mathbf{x}, \mathbf{p}) \in K \end{array}$$

is equivalent to the problem

$$\begin{array}{ll} \min_{\mathbf{x}, t} & t \\ \text{s.t.} & f(\mathbf{x}, \mathbf{p}) \leq t \\ & G(\mathbf{x}, \mathbf{p}) \in K. \end{array}$$

This last problem has all its uncertainties in its constraints only.

Objective Robustness

Another important robustness concept is objective robustness. This refers to solutions that will remain close to optimal for all possible realizations of the uncertain problem parameters. Since such solutions may be difficult to obtain, especially when uncertainty sets are relatively large, an alternative goal for objective robustness is to find solutions whose worst-case behavior is optimized. The worst-case behavior of a solution corresponds to the value of the objective function for the worst possible realization of the uncertain data for that particular solution. We now develop a mathematical model that addresses objective robustness. Consider an optimization problem of the form:

$$\min_{\mathbf{x} \in S} f(\mathbf{x}, \mathbf{p}).$$

Here, S is a (certain) feasible set and $f(x, p)$ is the objective function that depends on uncertain parameters p. As before, let $\mathcal{U}$ denote the uncertainty set that contains all possible values of the uncertain parameters p. Then an objective robust solution can be obtained by solving the *saddle-point problem*

$$\min_{x \in S} \max_{p \in \mathcal{U}} f(x, p).$$

As indicated at the end of the previous subsection, objective robustness can be seen as a special case of constraint robustness via a suitable reformulation. However, it is important to distinguish between these two problem variants as their "natural" robust formulations lead to two different classes of optimization formulations. Robust-constraint problems naturally lead to optimization problems with infinitely many constraints whereas robust-objective problems naturally lead to saddle-point problems. There are different methodologies available for each of these two problem classes.

Relative Robustness

The focus of constraint and objective robustness models on an absolute measure of worst-case performance is not consistent with risk tolerances of many decision makers. Instead, we may prefer to measure the worst case in a relative manner, relative to the best possible solution under each scenario. This leads us to the notion of relative robustness. Consider the optimization problem

$$\min_{x \in S} f(x, p), \tag{19.3}$$

where p is uncertain with uncertainty set $\mathcal{U}$. To simplify the description, we restrict our attention to the case with objective uncertainty and assume that the constraints are certain.

Given a fixed $\mathbf{p} \in \mathcal{U}$, let $z^*(\mathbf{p})$ denote the optimal value function

$$z^*(\mathbf{p}) := \min_{\mathbf{x} \in S} f(\mathbf{x}, \mathbf{p}).$$

Furthermore, define the optimal solution map

$$x^*(\mathbf{p}) = \arg\min_{\mathbf{x} \in S} f(\mathbf{x}, \mathbf{p}).$$

Note that $z^*(\mathbf{p})$ can be extended-valued and $\mathbf{x}^*(\mathbf{p})$ can be set-valued. To motivate the notion of relative robustness we first define a measure of regret associated with a decision after the uncertainty is resolved. If we choose $\mathbf{x}$ as our vector and $\mathbf{p}$ is the realized value of the uncertain parameter, the regret associated with choosing $\mathbf{x}$ instead of an element of $\mathbf{x}^*(\mathbf{p})$ is defined as

$$r(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}) - z^*(\mathbf{p}) = f(\mathbf{x}, \mathbf{p}) - f(x^*(\mathbf{p}), \mathbf{p}).$$

Note that the regret function is always non-negative and can also be regarded as a measure of the "benefit of hindsight". Now, for a given $\mathbf{x} \in S$ consider the maximum regret function:

$$R(\mathbf{x}) := \max_{\mathbf{p} \in \mathcal{U}} r(\mathbf{x}, \mathbf{p}) = \max_{\mathbf{p} \in \mathcal{U}} f(\mathbf{x}, \mathbf{p}) - z^*(\mathbf{p}).$$

A relative robust solution to problem (19.3) is a vector $\mathbf{x}$ that minimizes the maximum regret:

$$\min_{x \in S} \max_{p \in \mathcal{U}} f(x, p) - z^*(p). \tag{19.4}$$

Frame Title

While they are intuitively attractive, relative robust formulations can also be significantly more difficult than the standard absolute robust formulations. Indeed, since $z^*(\mathbf{p})$ is the optimal value function and involves an optimization problem itself, the problem (19.4) is a three-level optimization problem as opposed to the two-level problems in absolute robust formulations. Furthermore, the optimal value function $z^*(\mathbf{p})$ is rarely available in analytic form, is typically non-smooth, and is often hard to analyze. Another difficulty is that if f is linear in $\mathbf{p}$, as is often the case, then $z^*(\mathbf{p})$ is a concave function. Therefore, the inner maximization problem in (19.4) is a convex maximization problem and is difficult for most $\mathcal{U}$.

A simpler variant of (19.4) can be constructed by deciding on the maximum level of regret to be tolerated beforehand and by solving a feasibility problem instead with this level imposed as a constraint. For example, if we decide to limit the maximum regret to R, then the problem to solve becomes the following: find an $\mathbf{x}$ satisfying $\mathbf{x} \in S$ such that

$$f(\mathbf{x}, \mathbf{p}) - z^*(\mathbf{p}) \leq R, \text{ for all } \mathbf{p} \in \mathcal{U}.$$

If desired, one can then perform a bisection search on R to find its optimal value. Another variant of relative robustness models arises when we measure the regret in terms of the proximity of our chosen solution to the optimal solution set rather than in terms of the optimal objective values. For this model, consider the following distance function for a given $\mathbf{x}$ and $\mathbf{p}$:

$$d(\mathbf{x}, \mathbf{p}) = \inf_{x^* \in x^*(\mathbf{p})} \|\mathbf{x} - \mathbf{x}^*\|.$$

When the solution set is a singleton, there is no optimization involved in the definition. As above, we then consider the maximum distance function

$$D(\mathbf{x}) = \max_{\mathbf{p}\in\mathcal{U}} d(\mathbf{x},\mathbf{p}) = \max_{\mathbf{p}\in\mathcal{U}} \inf_{\mathbf{x}^*\in\mathbf{x}^*(\mathbf{p})} \|\mathbf{x}-\mathbf{x}^*\|.$$

For relative robustness in this new sense, we seek $\mathbf{x}$ that solves

$$\min_{\mathbf{x}\in S} \max_{\mathbf{p}\in\mathcal{U}} d(\mathbf{x},\mathbf{p}). \tag{19.5}$$

This variant is an attractive model for cases where we have time to revise our decision variables $\mathbf{x}$, perhaps only slightly, once $\mathbf{p}$ is revealed. In such cases, we will want to choose an $\mathbf{x}$ that will not need much perturbation under any scenario, i.e., we seek the solution to (19.5). This model can also be useful for multi-period problems where revisions of decisions between periods can be costly. Portfolio rebalancing problems with transaction costs are examples of such settings.

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In this section we review a few of the commonly used techniques for the solution of robust optimization problems. The tools we discuss are essentially reformulation strategies for robust optimization problems so that they can be rewritten as a deterministic optimization problem with no uncertainty. In these reformulations, we look for economy, so that the new formulation is not much bigger than the original, "uncertain" problem, and tractability, so that the new problem can be solved efficiently using standard optimization methods. The variety of the robustness models and the types of uncertainty sets rule out a unified approach. However, there are some common threads and the material in this section can be seen as a guide to the available tools which can be combined or appended with other techniques to solve a given problem in the robust optimization setting.

Sampling

One of the simplest strategies for achieving robustness under uncertainty is to sample several scenarios for the uncertain parameters from a set that contains possible values of these parameters. This sampling can be done with or without using distributional assumptions on the parameters and produces a robust optimization formulation with a finite uncertainty set. If uncertain parameters appear in the constraints, we create a copy of each such constraint corresponding to each scenario. Uncertainty in the objective function can be handled in a similar manner. Consider the generic uncertain optimization problem

$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}, \mathbf{p}) \\ \text{s.t.} & G(\mathbf{x}, \mathbf{p}) \in K, \text{ for all } \mathbf{p} \in \mathcal{U}. \end{array}$$

If the uncertainty set $\mathcal{U}$ is a finite set, i.e., $\mathcal{U} = \{p_1, p_2, \ldots, p_k\}$, the robust formulation is obtained as follows:

$$\begin{array}{ll} \min_{x,t} & t \\ \text{s.t.} & f(x, p_i) \leq t,\ i = 1, \ldots, k \\ & G(x, p_i) \in K,\ i = 1, \ldots, k. \end{array}$$

Note that no reformulation is necessary in this case and the duplicated constraints preserve the structural properties (linearity, convexity, etc.) of the original constraints. Consequently, when the uncertainty set is a finite set the resulting robust optimization problem is larger but theoretically no more difficult than the non-robust version of the problem. The situation is somewhat similar to stochastic programming formulations. Examples of robust optimization formulations with finite uncertainty sets can be found, for example in Rustem and Howe (2002).

Conic Optimization

Moving from finite uncertainty sets to continuous sets such as intervals or ellipsoids presents a theoretical challenge. The robust version of an uncertain constraint that has to be satisfied for all values of the uncertain parameters in a continuous set results in a semi-infinite optimization formulation. These problems are called semi-infinite since there are infinitely many constraints but only finitely many variables. Fortunately, for some types of uncertainty sets, it is possible to reformulate their robust semi-infinite programming versions using a finite set of conic constraints. To illustrate this, consider the following simple linear program:

$$
\begin{aligned}
\max_{\mathbf{x}} \quad & \mathbf{r}^\top \mathbf{x} \\
\text{s.t.} \quad & \mathbf{1}^\top \mathbf{x} = 1 \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned}
\tag{19.6}
$$

What is the optimal solution to this linear program? Now suppose the objective coefficients are uncertain with ellipsoidal uncertainty, e.g., suppose the objective coefficient vector $\mathbf{r}$ can be any element in the uncertainty set

$$\mathcal{U} = \{\mathbf{r} : \|\mathbf{r} - \boldsymbol{\mu}\|_2 \leq \delta\},$$

where $\boldsymbol{\mu}$ is the "nominal" value of $\mathbf{r}$. The robust version of (19.6) is

$$\begin{array}{ll} \max_{\mathbf{x}} & \min_{\mathbf{r}\in\mathcal{U}} \mathbf{r}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{1}^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Some simple calculations show that for a given $\mathbf{x}$

$$\min_{\mathbf{r}\in\mathcal{U}} \mathbf{r}^\top \mathbf{x} = \boldsymbol{\mu}^\top \mathbf{x} - \delta \cdot \|\mathbf{x}\|_2.$$

Thus the robust version of (19.6) is

$$\begin{array}{ll} \max_{\mathbf{x}} & \boldsymbol{\mu}^\top \mathbf{x} - \delta \cdot \|\mathbf{x}\|_2 \\ \text{s.t.} & \mathbf{1}^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

The latter problem can be rewritten as the following conic program:

$$\begin{array}{ll} \max_{x,t} & \boldsymbol{\mu}^\top \mathbf{x} - \delta \cdot t \\ \text{s.t.} & \mathbf{1}^\top \mathbf{x} = 1 \\ & \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \in \mathbb{L}_{n+1} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

More generally, suppose $\mathbf{r}$ has the following ellipsoidal uncertainty set:

$$\mathcal{U} = \{\mathbf{r} : (\mathbf{r} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{r} - \boldsymbol{\mu}) \leq \delta^2\}$$

for some symmetric and positive definite matrix Σ. Then the robust version of (19.6) is

$$\begin{array}{ll} \max & \boldsymbol{\mu}^\top \mathbf{x} - \delta \cdot \sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} \\ \text{s.t.} & \mathbf{1}^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

which again can be formulated as a conic program. Observe the resemblance between the latter model and Markowitz's mean-variance model.

The machinery of robust optimization can be applied to mean-variance portfolio optimization to mitigate the effects of estimation errors in the expected returns and/or in the covariance matrix (Ceria and Stubbs, 2006; Goldfarb and Iyengar, 2003). The basic idea is to consider the mean-variance optimization problem in one of its forms, e.g.,

$$\begin{array}{ll} \max_{\mathbf{x}} & \boldsymbol{\mu}^\top \mathbf{x} - \frac{1}{2}\gamma \cdot \mathbf{x}^\top \mathbf{V}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{C}\mathbf{x} \leq \mathbf{d}, \end{array} \tag{19.7}$$

and assume $\boldsymbol{\mu}$ belongs to some uncertainty set,

$$\mathcal{U} = \{\boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \Sigma^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \delta^2\}.$$

Then the robust version of (19.7) is

$$
\begin{array}{ll}
\max_{\mathbf{x}} & \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \delta \cdot \sqrt{\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x}} - \frac{1}{2}\gamma \cdot \mathbf{x}^\top \mathbf{V} \mathbf{x} \\
\text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{C}\mathbf{x} \leq \mathbf{d}.
\end{array}
\tag{19.8}
$$

We next show that (19.8) is a conic program. To that end, it suffices to find a conic representation of the objective function. Let $\mathbf{R} \in \mathbb{R}^{n \times p}$, $\mathbf{L} \in \mathbb{R}^{n \times q}$ be such that $\boldsymbol{\Sigma} = \mathbf{R}\mathbf{R}^\top$ and $\mathbf{V} = \mathbf{L}\mathbf{L}^\top$. Both $\mathbf{R}$ and $\mathbf{L}$ exist because $\boldsymbol{\Sigma}$ and $\mathbf{V}$ are positive semidefinite.

By introducing new variables $\mathbf{s}, \mathbf{t}$, the problem (19.8) can be rewritten as the following conic program:

$$
\begin{aligned}
\max_{\mathbf{x}, s, t} \quad & \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \delta \cdot s - \frac{1}{2}\gamma \cdot t \\
\text{s.t.} \quad & \begin{bmatrix} s \\ \mathbf{Rx} \end{bmatrix} \in \mathbb{L}_{p+1} \\
& \begin{bmatrix} t+1 \\ t-1 \\ 2\mathbf{Lx} \end{bmatrix} \in \mathbb{L}_{q+2} \\
& \mathbf{Ax} = \mathbf{b} \\
& \mathbf{Cx} \leq \mathbf{d}.
\end{aligned}
$$

Saddle-Point Characterizations

For the solution of problems arising from objective uncertainty, the robust solution can be characterized using saddle-point conditions when the original problem satisfies certain convexity assumptions. The benefit of this characterization is that we can then use algorithms such as interior-point methods already developed and available for saddle-point problems. As an example of this strategy, consider the objective-robust formulation discussed in Section 19.2:

$$\min_{\mathbf{x}\in S}\max_{\mathbf{p}\in U} f(\mathbf{x},\mathbf{p}). \tag{19.9}$$

We note that the dual of this robust optimization problem is obtained by changing the order of the minimization and maximization problems:

$$\max_{\mathbf{p}\in\mathcal{U}} \min_{\mathbf{x}\in S} f(\mathbf{x}, \mathbf{p}). \tag{19.10}$$

Under mild assumptions on $f, S, \mathcal{U}$, there exists a saddle-point solution $(\mathbf{x}^*, \mathbf{p}^*) \in S \times \mathcal{U}$ such that

$$f(\mathbf{x}^*, \mathbf{p}) \leq f(\mathbf{x}^*, \mathbf{p}^*) \leq f(\mathbf{x}, \mathbf{p}^*) \text{ for all } \mathbf{x} \in S, \mathbf{p} \in \mathcal{U}.$$

This characterization is the basis of the robust optimization algorithms given in Tütüncü and Koenig (2004).

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Since many financial optimization problems involve future values of security prices, interest rates, exchange rates, etc., which are not known in advance but can only be forecasted or estimated, such problems fit perfectly into the framework of robust optimization. We next describe some examples of robust optimization formulations for a variety of financial optimization problems.

Robust Profit Opportunities in Risky Portfolios

Consider an investment environment with n financial securities whose future price vector $\mathbf{r} \in \mathbb{R}^n$ is a random variable. Let $\mathbf{p} \in \mathbb{R}^n$ represent the current prices of these securities. If the investor chooses a portfolio $\mathbf{x} = [x_1 \ \cdots \ x_n]$ that satisfies

$$\mathbf{p}^\top \mathbf{x} < 0$$

and the realization of the random variable $\mathbf{r}$ satisfies

$$\mathbf{r}^\top \mathbf{x} \geq 0 \tag{19.11}$$

then there is an arbitrage opportunity: an investor could make money by constructing the portfolio $\mathbf{x}$ with negative cash flow ($\mathbf{p}^\top \mathbf{x}$ is negative, pocketing money) and subsequently collecting the non-negative cash flow $\mathbf{r}^\top \mathbf{x}$ of the portfolio $\mathbf{x}$.

Since arbitrage opportunities generally do not persist in financial markets, one might be interested in the alternative and weaker profitability notion where the non-negativity of the portfolio is only guaranteed to occur with high probability. More precisely, consider the following relaxation of (19.11):

$$\mathrm{P}(\mathbf{r}^\top \mathbf{x} \geq 0) \geq 0.99. \tag{19.12}$$

Let μ and Q represent the expected future price vector and covariance matrix of the random vector r. Then $E(r) = \mu^\top x$ and $\text{stdev}(r^\top x) = \sqrt{x^\top Qx}$. If the random vector r is Gaussian, then (19.12) is equivalent to

$$\mu^\top x - \theta \cdot \sqrt{x^\top Qx} \geq 0,$$

where $\theta = \Phi^{-1}(0.99)$ and Φ is the standard normal cumulative distribution.

Therefore, if we find an $\mathbf{x}$ satisfying

$$\mu^\top \mathbf{x} - \theta \cdot \sqrt{\mathbf{x}^\top Q \mathbf{x}} \geq 0, \quad \mathbf{p}^\top \mathbf{x} < 0,$$

for a large enough positive value of θ, we have an approximation of an arbitrage opportunity. Note that, by relaxing the constraint $\mathbf{p}^\top \mathbf{x} < 0$ as $\mathbf{p}^\top \mathbf{x} \leq 0$ or as $\mathbf{p}^\top \mathbf{x} \leq -\varepsilon$, we obtain a conic feasibility system. Therefore, the resulting system can be solved using the conic optimization approaches.

We next explore some portfolio selection models that incorporate the uncertainty of problem inputs.

Robust Portfolio Selection

This section is adapted from Tütüncü and Koenig (2004). Recall that Markowitz's mean-variance optimization problem can be stated in the following form, which combines the reward and risk in the objective function:

$$\max_{\mathbf{x} \in \mathcal{X}} \mu^\top \mathbf{x} - \frac{\gamma}{2} \cdot \mathbf{x}^\top \mathbf{Q} \mathbf{x}. \tag{19.13}$$

Here μ and $\mathbf{Q}$ are respectively estimates of the vector of expected values and covariance of returns of a universe of securities, and γ is a risk-aversion constant used to trade off the reward (expected return) and risk (portfolio variance).

The set $\mathcal{X}$ is the set of feasible portfolios which may carry information on short-sale restrictions, sector distribution requirements, etc. Since such restrictions are predetermined, we can assume that the set $\mathcal{X}$ is known without any uncertainty at the time the problem is solved.
Recall also that solving the problem above for different values of γ we obtain the *efficient frontier* of the set of feasible portfolios. The optimal portfolio will be different for individuals with different risk-taking tendencies, but it will always be on the efficient frontier.

One of the limitations of this model is its need to accurately estimate the expected returns and covariances. In Bawa et al. (1979), the authors argue that using estimates of the unknown expected returns and covariances leads to an *estimation risk* in portfolio choice, and that methods for optimal selection of portfolios must take this risk into account. Furthermore, the optimal solution is sensitive to perturbations in these input parameters - a small change in the estimate of the return or the variance may lead to a large change in the corresponding solution; see, for example, Michaud and Michaud (2008). This attribute is unfavorable since the modeler may want to periodically rebalance the portfolio based on new data and may incur significant transaction costs to do so. Furthermore, using point estimates of the expected return and covariance parameters does not fulfill the needs of a conservative investor. Such an investor would not necessarily trust these estimates and would be more comfortable choosing a portfolio that will perform well under a number of different scenarios.

Of course, such an investor cannot expect to get better performance on some of the more likely scenarios, but will have insurance for more extreme cases. All these arguments point to the need of a portfolio optimization formulation that incorporates robustness and tries to find a solution that is relatively insensitive to inaccuracies in the input data. Since all the uncertainty is in the objective function coefficients, we seek an objective robust portfolio, as outlined in Section 19.2.

For *robust portfolio* optimization we consider a model that allows covariance matrix information to be given in the form of intervals. For example, this information may take the form "the expected return on security j is between 8% and 10%" rather than claiming that it is 9%. Mathematically, we will represent this information as membership in the following set:

$$\mathcal{U} = \left\{ (\mu, Q) : \mu^L \leq \mu \leq \mu^U, \quad Q^L \leq Q \leq Q^U, \quad Q \succeq 0 \right\}, \tag{19.14}$$

where μ^L, μ^U, Q^L, Q^U are the extreme values of the intervals we just mentioned.

The restriction $Q \succeq 0$ is necessary since Q is a covariance matrix and, therefore, must be positive semidefinite. These intervals may be generated in different ways. An extremely cautious modeler may want to use historical lows and highs of certain input parameters as the range of their values. One may generate different estimates using different scenarios on the general economy and then combine the resulting estimates. Different analysts may produce different estimates for these parameters and one may choose the extreme estimates as the endpoints of the intervals intervals. One may choose a confidence level and then generate estimates of covariance and return parameters in the form of prediction intervals.

We want to find a portfolio that maximizes the objective function in (19.13) in the worst-case realization of the input parameters μ and Q from their uncertainty set $\mathcal{U}$ in (19.14). Given these considerations the robust optimization problem takes the following form

$$\max_{\mathbf{x} \in \mathcal{X}} \min_{(\mu, Q) \in \mathcal{U}} \mu^\top \mathbf{x} - \frac{\gamma}{2} \cdot \mathbf{x}^\top Q \mathbf{x}. \tag{19.15}$$

This problem can be expressed as a saddle-point problem and be solved using the technique outlined in Halldórsson and Tütüncü (2003).

We consider the following simple three-asset portfolio model from Ceria and Stubbs (2006):

$$\begin{array}{ll} \max & \mu^\top \mathbf{x} \\ \text{s.t.} & TE(\mathbf{x}) \le 0.1 \\ & \mathbf{1}^\top \mathbf{x} = 1 \\ & \mathbf{x} \ge \mathbf{0}, \end{array} \tag{19.16}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ and

$$TE(\mathbf{x}) = \sqrt{\begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.5 \\ x_3 \end{bmatrix}^\top \begin{bmatrix} 0.1764 & 0.09702 & 0 \\ 0.9702 & 0.1089 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.5 \\ x_3 \end{bmatrix}}.$$

This is essentially a two-asset portfolio optimization problem where the third asset represents the proportion of the funds that are not invested. The first two assets have standard deviations of 42% and 33% respectively and a correlation coefficient of 0.7. The "benchmark" is the portfolio that invests funds half-and-half in the two assets. The function $TE(\mathbf{x})$ represents the tracking error of the portfolio with respect to the half-and-half benchmark and the first constraint indicates that this tracking error should not exceed 10%. The second constraint is the budget constraint; the third enforces no shorting. We depict the projection of the feasible set of this problem onto the space spanned by variables x_1 and x_2 in Figure 19.1.

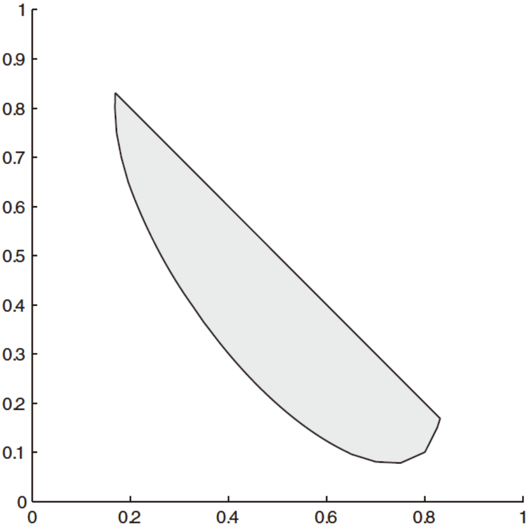

Figure 19.1: The feasible set of the mean–variance model (19.16)

We now build a relative robustness model for this portfolio problem. We assume that the covariance matrix estimate is certain. We consider a very simple uncertainty set for the expected return estimates consisting of three scenarios represented with the three arrows in Figure 19.2. These three scenarios correspond to the following values for μ: $(6, 4, 0)$, $(5, 5, 0)$, and $(4, 6, 0)$. When $\mu = (6, 4, 0)$ the optimal solution is $(0.831, 0.169, 0)$ with objective value 5.662. Similarly, when $\mu = (4, 6, 0)$ the optimal solution is $(0.169, 0.831, 0)$ with objective value 5.662. When $\mu = (5, 5, 0)$ all points between the previous two optimal solutions are optimal solutions are optimal with a shared objective value of 5.0.

Therefore, the relative robust formulation for this problem can be written as follows:

$$
\begin{aligned}
\min_{\mathbf{x},t} \quad & t \\
\text{s.t.} \quad & 5.662 - (6x_1 + 4x_2) \le t, \\
& 5.662 - (4x_1 + 6x_2) \le t, \\
& 5 - (5x_1 + 5x_2) \le t, \\
& TE(\mathbf{x}) \le 0.1, \\
& \mathbf{1}^\top \mathbf{x} = 1, \\
& \mathbf{x} \ge 0.
\end{aligned} \tag{19.17}
$$

Instead of solving the problem where the optimal regret level is a variable t in the formulation, an easier strategy is to choose a level of regret that can be tolerated and find portfolios that do not exceed this level of regret in any scenario. For example, choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

$$\begin{aligned}
&5.662 - (6x_1 + 4x_2) \le 0.75, \\
&5.662 - (4x_1 + 6x_2) \le 0.75, \\
&5 - (5x_1 + 5x_2) \le 0.75, \\
&TE(\mathbf{x}) \le 0.1, \\
&\mathbf{1}^\top \mathbf{x} = 1, \\
&\mathbf{x} \ge 0.
\end{aligned}$$

This problem and its feasible set of solutions is illustrated in Figure 19.2.

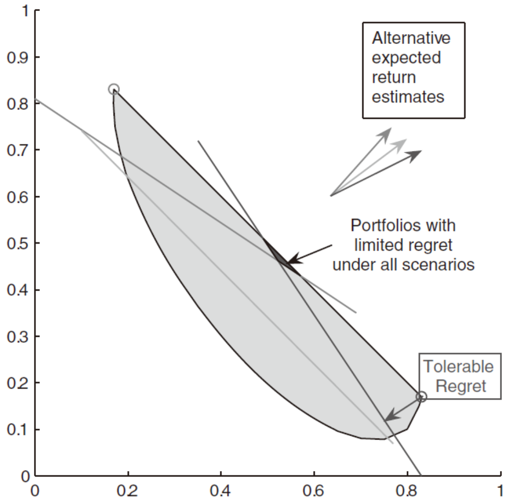

Figure 19.2: Set of solutions with regret less than 0.75 for the mean–variance model (19.16)

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Robust optimization was introduced by Ben-Tal and Nemirovski (1998, 2002) and independently by El Ghaoui and Lebret (1997) and El Ghaoui et al. (1998). The textbook by Ben-Tal et al. (2009) gives a thorough discussion on the subject, including an extensive list of references. Although robust optimization is widely popular in a variety of disciplines, it is not as widespread in financial optimization yet. There are strong supporters of its potential in finance (Ceria and Stubbs, 2006; Goldfarb and Iyengar, 2003; Tütüncü and Koenig, 2004). There are also some skeptics (Scherer, 2007).