第二章

1. $f(x)=2x^7+5x^5+24x^4+33x^2-127x+58$ ，求：

$f[1,2,4,8,16,32,64,128]$； $f[1,2,4,8,16,32,64,128,256]$

答案：

$$f[x_0,x_1,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!},\quad \xi\in[\min\{x_0,x_1,\cdots,x_n\},\max\{x_0,x_1,\cdots,x_n\}]$$

$$f[1,2,4,8,16,32,64,128]=\frac{f^{(7)}(\xi)}{7!}=\frac{2\times 7!}{7!}=2$$

$$f[1,2,4,8,16,32,64,128,256]=\frac{f^{(8)}(\xi)}{8!}=\frac{0}{8!}=0$$

2. 设分段多项式

$$S(x)=\begin{cases}x^3+x^2,0\le x\le 1\\2x^3+bx^2+cx-1,1\le x\le 2\end{cases}$$

是以$x=0,1,2$为节点的三次样条函数，试确定系数b和c的值.

答案：

$S(1)=2+b+c-1=2$ ，故$b+c=1$

$$S'(x)=\begin{cases}3x^2+2x,0\le x\le 1\\6x^2+2bx+c,1\le x\le 2\end{cases}$$

$S'(1)=6+2b+c=3+2=5$，故$2b+c=-1$

所以$b=-2$， $c=3$

3. 设$f(x)=(x-x_0)(x-x_1)\cdots(x-x_n)$，试求$f[x_0,x_1,\cdots,x_p]$的值， $p\le n+1$

答案：

$$f(x)=(x-x_0)(x-x_1)\cdots(x-x_n)=w_{n+1}(x)$$

$f(x_i)=0$，$(i=0,1,...,n)$

$$f[x_0,x_1,\cdots,x_p]=\sum_{j=0}^{p}\frac{f(x_j)}{w_p'(x_j)}$$

当 $p \le n$ 时，有 $f[x_0, x_1, \cdots, x_p] = 0$

当 $p = n+1$ 时，有 $f[x_0, x_1, \cdots, x_{n+1}] = \sum_{j=0}^{n+1} \frac{f(x_j)}{w'_{n+2}(x_j)} = \frac{f(x_{n+1})}{f(x_{n+1})} = 1$

综上 $f[x_0, x_1, \cdots, x_p] = \begin{cases} 0, p \le n \\ 1, p = n+1 \end{cases}$

4. 若 $f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$ 有 n 个不同实根 $x_1, x_2, \cdots, x_n$，求证：

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, 0 \le k \le n-2 \\ \frac{1}{a_n}, k = n-1 \end{cases}$$

答案：

由 $f(x)$ 是 n 次多项式且有互异实根 $\{x_i\}$，$(i = 1, 2, \cdots, n)$ 可知

$$f(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_n) = a_n w_n(x)$$

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \sum_{j=1}^{n} \frac{x_j^k}{a_n w'_n(x_j)} = \frac{1}{a_n} \sum_{j=1}^{n} \frac{x_j^k}{w'_n(x_j)}$$

记 $g(x) = x^k$，并利用差商的函数值表达形式有

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \frac{1}{a_n} \sum_{j=1}^{n} \frac{g(x_j)}{w'_n(x_j)} = \frac{1}{a_n} g[x_1, x_2, \cdots, x_n]$$

再由差商和导数的关系知

$$\sum_{j=1}^{n} \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, 0 \le k \le n-2 \\ \frac{1}{a_n} \frac{g^{(n-1)}(\xi)}{(n-1)!} = \frac{1}{a_n}, k = n-1 \end{cases}$$